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A DESIGN PROCEDURE BASED ON THE QUADRATIC PERFORMANCE INDEX AND LINEAR LEAST SQUARES APPROXIMATIONS

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SUMMARY

A package of known design techniques are assembled as a design procedure for automatic control systems, and the problems of applying the procedure to a class of plants is empirically examined. Key elements in the procedure consist of the weighted quadratic performance index, Bell's method (1) of weight selection, and the use of linear least squares to construct the closed-loop feedback functions.

The procedure is intended for design problems delimited by several rather broad characteristics. Thus, response requirements are assumed ameanable to the use of the weighted quadratic performance index $J = \int_0^T (x^T Qx + u^2)dt$ (e.g., settling time, overshoot limits, specified response envelopes). Secondly, the desired closed-loop control function must be of the form $\hat{u} = \sum^r k_i Z_i(x,t)$, linear in a set of gain parameters k_i. Note the functions Z_i may be linear or nonlinear in the state and time variables x,t.

Applications in the study are limited to one class of plants. The class, defined partly in terms of a block diagram arrangement, consists of two unilateral elements in cascade, at least one of which can be written as a set of linear system dynamics. The problem considered is the so-called regulator problem.

The design procedure consists of five basic steps, as follows: (1) Select amplitude weights Q, using Bell's method, and a response time interval T. (2) Compute open-loop control data using Pontryagin's

necessary conditions for minimizing the cost index J. Revise the numbers in step one if specified performance is not achieved. The final two point boundary solution data is then stored on magnetic tape. (3) Select the functions Z_i to be used for the closed-loop control. Hardware constraints and, if desired, step-up procedures are aids in the selection. (4) Compute the closed-loop gains k_i by a least squares fit to the stored open-loop data. Several candidate gain sets may be rapidly computed, based upon alternate Z_i functions and data point selections. (5) Check closed-loop performance of candidate controls by simulation. A trade-off between simulation performance and hardware simplicity determines the design selection.

The empirical study is defined and carried out in Chapter IV. It consists of a performance analysis under the variation of five basic types of independent empirical variables, viz., plant, cost index, basis (control) function, open-loop data base, and initial condition variations. Since performance requirements may vary widely with applications, any selection of performance criteria for the present study must be somewhat arbitrary. In this study, performance is revealed by trajectory plots as well as cost index figures, terminal error measures, and several response time figures including rise time and settling time. Though not all inclusive, these measures do allow valid comparisons for extrapolation of the results to the needs of individual design problems. For convenience, Table 1 of Chapter IV summarizes the combinations of empirical variations which are tested.

The experimental results are developed through the application of the procedure to three case studies. The first of these consists of some second order plant dynamics in cascade with a linear gain element.

As such, the true feedback solution is known in closed form, thus permitting the design method to first be tested with this insight. The results reveal that in terms of closed-loop performance, success of the least squares fitting is rather sensitive to a proper choice of the Z_i functions. A step-up procedure, operating from a least squares criterion as discussed in Appendix E, was shown to yield closed-loop efficient Z_i selections. Thus, the least squares criterion appears to permit efficient discrimination between candidate Z_i selections for closed-loop controls, even though a high level of least squares correlation was in itself a poor predictor of success.

The second case is a sequence of four free response time problems, consisting of the same second order plant dynamics in cascade with a gain element whose shape is varied in turn from the linear case to successively greater degrees of nonlinearity. The flooding technique was especially useful in the efficient solution of the two point boundary problem open-loop data of this case. As in the first case study, efficient closed-loop functions were found for each of the four gain shape subcases.

The third case considers higher order plant dynamics through the study of a fourth order servo with nonlinear two phase control motors. This case is useful in revealing practical restrictions on the application of the method. These restrictions arise from computational problems with the two point boundary solutions for the open-loop data. Thus, extreme co-state sensitivities limited solutions to short response time selections. At added cost, this difficulty could be eased by resorting to double precision arithmetic. Secondly, however, slow boundary value convergence rates made computation costs prohibitive in the nonlinear problem, for all but moderately small initial conditions. Moreover, using the flooding technique as an alternate to the adjoint method, as was done in the second case problem, did not prove feasible because of the high co-state sensitivities. Thus, full application of the design procedure was restricted in the third problem to linearized models of the plant. Unfortunately, these added restrictions are apparently very problem dependent. Hence, existence of the limitations in a given design problem are not readily established a priori.

Regarding the amplitude weight selections Q, it is of interest to note that the method and associated tables given by Bell may lead to a nonconvexity condition of the cost index, thereby failing to meet a common sufficiency requirement for a minimizing solution. Convexity is satisfied when the weighting matrix Q is positive semi-definite. Since other choices are shown to give useful candidate responses, the traditional emphasis on nonnegative Q appears to be overly restrictive when the role of the index is viewed as a response shaping device. In the third case study the co-state sensitivity problem is shown to exert an overriding influence on the weight selection choice.

CHAPTER I

INTRODUCTION

Statement of Objective

The primary objective of this investigation is to conduct an empirical study of a design procedure for automatic control systems with a class of plant. The design procedure utilizes the weighted quadratic integral performance index to constrain the formulation of candidate designs, obtained in the form of open-loop solutions. Linear least squares methods are then employed to construct selected closed-loop feedback functions which approximate the constrained open-loop control solutions. The role of the weighted index is viewed as a response shaping device.

It is intended that the work result in an accumulation of empirical experience with both the overall procedure and points of interest regarding its component parts. In the latter group, chief interests lie in the quadratic index as a design constraint, its application to the class of plants, Bell's weight selection method (1), the use of linear least squares feedback laws, and computational aspects of the problem. From this experience it is hoped that some judgements could be made regarding design costs and prospects of success with similar problems, and to afford a clear understanding of the limitations of the method.

Background and History

(i) General Nature of the Design Problem

The design problem begins with a set of requirements. These consist of performance specifications and design constraints, both of which are here postulated to vary with the application. In automatic control work the plant (controlled process) is a fundamental design constraint. Its relation to the controller, to be prescribed by the design procedure, is illustrated in the elementary block diagram of Figure 1.

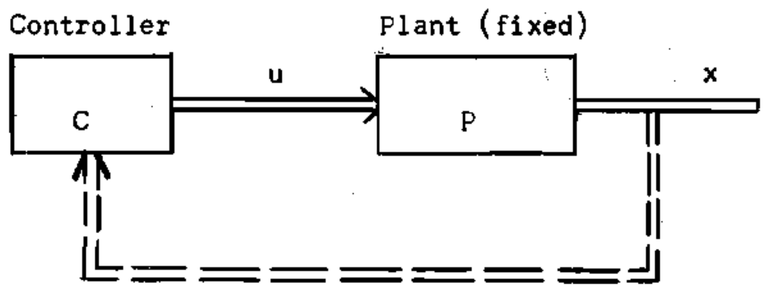

Figure 1. Elementary Block Diagram.

For best performance in the face of ever present disturbances the controller is structured in the form of a feedback controller, as shown.

Thus, the task of the design procedure, as here considered, is to synthesize a suitable controller relation $u = C(x,t)$ called the feedback control law such that all design requirements are met.

It is the nature of typical design requirements, as postulated, that the design solution is usually far from unique. Hence, the design procedure of interest here is characterized more by its potential suitability to various design requirements (e.g., prescribed response envelopes) rather than the ability to produce certain highly specialized types of solution (e.g., minimum response time). Thus, the role of the weighted quadratic index as a response shaping device is stressed in the work.

(ii) Background to Considered Design Method

As stated, the design method consists of two essential parts: (1) the use of the weighted quadratic integral as a design selection constraint and (2) the application of linear least squares to synthesize a feedback law from the open-loop control data. The purpose of this section is to give some general background to the use of these elements as a design procedure and to comment on the class of plant to which they will be applied. Precise definitions are reserved for later chapters.

Consider first the weighted quadratic integral performance index

$$J(u) = \int_0^T (x^T Q x + u^2) dt \tag{1.1}$$

The quadratic constraint is applied to the design problem through the use of Pontryagin's necessary conditions for minimizing the index. Thus, the solutions are referred to as optimal solutions. Conditions sufficient to insure that the necessary conditions yield a minimizing solution can be established. It turns out through experience, however, that the quadratic constraint may sometimes provide useful candidate design solutions under conditions where optimality is not assured. Therefore, in the remaining parts of this chapter the use of the term optimal is given the more loose interpretation based only on necessary conditions.

Strong interest in the quadratic cost index and its variations have made it near classic in the literature. Extensive results with linear plants were presented by Kalman, e.g. (18, 19), including closed form solutions and special computational procedures. Tyler and Tuteur (14) consider the multivariable control problem. Similarly Rynaski and and Whitbeck (20) extend the frequency domain approach of Chang (4) to linear, multivariable systems. Wonham and Johnson (15), and later Bass and Webber (16) apply the quadratic integral to bang-bang systems. In a Ph.D. thesis Bell (1) studied this index, and considered the effect of weight selections on response. Thus, selection of the index for the present study represents a continuation of effort within the School of Mechanical Engineering on the use of the index as a design tool. In the present study the use of Bell's weight selection method as a motivation for the continued choice of the index over other weighted cost functions is stressed. In addition, the use of the least squares approach is explored as a means for synthesizing the feedback function.

A further advantage to the selection of this index is that solutions can be obtained in closed form for the case of linear plants. This means that to a first order approximation (when such exists) the solutions for nonlinear plants are known as well. From the experience recorded in the literature (e.g., Ref. (1,5,13, 20)) it appears that the adjustable weights give the index adequate flexibility for many applications.

Consider next the second essential element of the design procedure, the use of linear least squares to synthesize the feedback law. The need for this second step is related to fundamental difficulties with the first step.

To motivate the choice of this second element, consider the nature of these difficulties. Pontryagin solutions of the cost function optimization are generally numerical results from differential equation solutions in the form of state and control time histories. As such, these results represent open-loop trajectories which give a limited tabulation of the unknown closed-loop feedback law. Thus, the difficulties begin by introducing two basic problems: (1) undesirable structure (open-loop time histories rather than the needed closed-loop control law), and (2) undesirable form of presentation (tabular data as opposed to the needed equation). In principle, the first objection can be overcome by using the Dynamic Programming approach of Bellman (2). Unfortunately the method essentially requires the solution of partial differential equations and still suffers from computational considerations in problems of even moderate dimensionality.

A third difficulty is that even if known, the exact feedback law would often be rejected for the final design, due to the probable expense of the necessary hardware. As even simple examples show, the exact solution may involve such normally unwanted terms as time varying hyperbolic sines and cosines.[1] This difficulty reflects the near impossibility for an otherwise convenient cost index to reflect the full range of design requirements, i.e., in this case hardware costs.

Thus, some modification of the direct application of the cost index variational approach is needed. One useful modification could be to choose the form of the feedback control law in advance, leaving a set of arbitrary parameters for adjustment. This constraint allows a direct

[1]See e.g. Ref. 1, Chapter III.

incorporation of hardware considerations, including the problem of limited state measurement. When this control function constraint is added, the cost optimization changes from a problem in variational calculus to one of ordinary calculus. This approach, referred to as parameter optimization, can lead to severe computational problems with the associated non-linear algebra on higher order plants.[1]

An alternative approach is to approximate the open-loop <u>control surface</u> by fitting selected surfaces to it as "closely" as one wishes (or is able). This approach is the one investigated here, and thus employs two steps in finding the parameter solutions. In the first step the open-loop control surface is generated and then permanently retained for use in the second step, where trial fittings of selected control functions can be carried out without repeating the basic open-loop calculations. Moreover, the fitting process can be carried out directly from the open-loop tabular data, without any distinct need to know or visualize the true surface. A disadvantage is that fitting a control surface does not assure a good fit in the sense of the cost criterion J.

The present work uses linear least squares as the criterion for fitting the control surfaces to the tabulated data. By linear it is meant that the class of control surface is any function of the state variables which is <u>linear in a set of arbitrary parameters</u> i.e., if $\hat{u}$ is to be the control then

$$\hat{u} = \sum_{i=1}^{r} k_i Z_i(x_1, x_2, \ldots, x_n) \tag{1.2}$$

[1]Recently, Howerton and Hammond (Ref. (21)) have reported successful results on a computational scheme for this problem with linear stationary plants.

where the Z_i are suitable linearly independent functions of x and t. The advantage of linear least squares is that a unique solution is guaranteed to exist and it is relatively straightforward to obtain computationally. A larger class of least squares, nonlinear in the parameters and far more difficult to solve, is thus specifically excluded. The linear (in the parameter) class of functions includes the more simple control functions for use in practical designs. In such cases where the solution truly requires more difficult functions for reasonable approximation, the real nature of the problem is most likely a failure of the cost index to reflect all the design requirements (i.e., hardware constraints) as discussed previously.

Much of the effort in least squares applications to open-loop control functions has originated with work sponsored by NASA Huntsville on synthesis of optimal guidance polynomials for the Saturn vehicle. In this work a minimum fuel flight path was the criterion for optimality. In one of the first papers to come from this work, Schmieder and Braud (23) referred to the optimal guidance as a path adaptive mode, and the least squares implementation as approximating a statistical model. Later Vance (24) referred to this as optimal adaptive guidance. After several years the bulk of the guidance work was summarized in a paper by Hauserman (25), who compared the path adaptive mode to a competing iterative (non-optimal) guidance scheme.

In other references to least squares control surfaces, Kipiniak (26) suggested the method as a way of constructing a feedback law in an optimal chemical process example. However, no actual results were presented. In the paper by Fisher (13), least squares methods were used to mechanize a "best" approximation to the linear control under conditions of limited state measurement. In a Ph.D. thesis, Hove (27) used least squares to construct control functions for selected trajectories. All results were run from a single initial condition and tested against a single trajectory, thus failing to reveal much about the success of the fittings as feedback laws. In a recent paper Smith (28) used least squares to fit a special class of segmented control function to time optimal bang-bang switching surfaces. The results were concluded to be generally good, with response times running on the order of 1 1/2 - 2 times that of the true time optimal case.

Before summarizing the design method in terms of a definite sequential step process, a brief description is given of the considered class of plant. As shown in Figure 2, the plant, defined partly in terms of a block diagram arrangement, is separable into a cascade arrangement of two unilateral elements, at least one of which can be written in terms of linear system dynamics.

Figure 2. Considered Class of Plant.

A more detailed description and precise definitions are given in Chapter II, where two subcases are recognized, depending in part upon which of the two blocks contains the linear dynamics.

The description of the design procedure is now summarized by defining the method in terms of a five step sequence.

Design Method Steps:

1. Select Q and T in (1.1).
2. Compute typical open-loop solutions u^*, x^* (Chapter II).
3. Select a form $\hat{u}$ from the admissible controls (1.2).
4. Compute the least squares solution $\hat{k}$ (Chapter III).
5. Simulate the design by computing $x[\hat{u}(\hat{k})]$ from selected initial conditions x_o.

Note that a design is actually specified after the first 4 steps, but as will be shown, the design cannot be adequately assessed until the final step of the method is carried out.

Scope of the Investigation

The objective has been stated as the investigation of a design method. In this chapter the method has been described and a general background given. In Chapters II and III some definitions are stated and the implementation of the five steps of the method are considered in some detail. The main content of the investigation consists of an empirical testing and evaluation of the design method (Chapter IV). The testing aspect implies that a part of the scope of the work must be concerned with an identification of relevant variables. A summary of the factors specifically considered is given in the first part of Chapter IV, prior to presenting the empirical case studies.

The evaluation aspect of the investigation implies a need for an evaluation criterion. In the actual design situation a criterion is supplied directly by the particular design requirements. Since requirements will vary with the application, however, a part of the scope of the present work must be concerned with setting up an adequate criteria for comparing the results presented. The selected criteria are summarized in the first part of Chapter IV. In Chapter V the results are discussed and conclusions presented.

Since the work largely involves an empirical testing process, this thesis could be described as experimental, rather than analytical in nature. By providing the all important means of implementing the design algorithms and simulating the design results, the extensive role of the computer in the work is therefore somewhat analogous to an experimental test set-up.

CHAPTER II

OPEN-LOOP TRAJECTORIES

Plant and Cost Constraints

This chapter considers the first two steps of the design procedure, viz., the generation of open-loop data for use in the closed-loop constructions. Relevant theory is sketched and essential points of the implementation are revealed. The use of Bell's cost index weight selection method is discussed.

As a preliminary step the plant and cost index constraints are first given more precise definitions.

Plant: The plant is defined for two types of unilateral cascade block elements, at least one of which is linear, as follows:

Type 1. A nonlinear gain element in cascade with linear dynamics.

$$\frac{dx}{dt} = Ax + g(u) \qquad x(0) = x_o$$
$$x(T) = 0$$

Type 2. A linear dynamic element in cascade with nonlinear dynamics.

$$\frac{dx}{dt} = Ax + f(x) + bu \qquad x(0) = x_o$$
$$x(T) = 0$$

where

$x \in E^n$ = plant state vector.
$u \in \Omega$ = plant control scalar.
Ω = range of u.
A = a constant nxn matrix with nonpositive real eigenvalues.
b = a constant vector.
f = a continuous, bounded, nonlinear, vector function of x having continuous first and second derivatives in x.
g = a continuous, bounded, linear or nonlinear vector function of u having continuous first derivatives in u.
$t \in [0,T]$ = time.

A third possible type, a nonlinear dynamic element in cascade with linear dynamics ($\dot{x} = Ax + f(x,u)$), is not considered in any of the work to follow.

Cost Index: The cost index constraint J is taken to be the quadratic functional

$$J(u) = \int_0^T (x^T Q x + u^2)dt \tag{2.1}$$

where

Q = a constant, symmetric, nxn weighting matrix.
T = a selected fixed or free response interval.

In some of the analysis to follow, it is convenient to express the two types of plant differential equations by the single composite relation

$$\dot{x} = Ax + f(x) + g(u) \qquad \begin{aligned} x(0) &= x_o \\ x(T) &= 0 \end{aligned} \tag{2.2}$$

$$g(u) = bu \quad \text{if } f(x) \neq 0, \quad \text{any } t \in [0,t]$$

In addition, the cost index is temporarily expressed in the form

$$J(u) = \int_0^T (F(x) + G(u))dt \tag{2.3}$$

In this form the role of the quadratic assumption in the development is more readily apparent.

Open-Loop Formulation

The analysis proceeds from the formation of the Hamiltonian scalar function

$$H = -F(x) - G(u) + p^T(Ax + f(x) + g(u)) \tag{2.4}$$

where $p \varepsilon E^n$ is a continuous vector time function called the co-state. Equation (2.4) then yields the differential system

$$\begin{aligned} \dot{x} &= \nabla_p H = Ax + f(x) + g(u) & x(0) &= x_o \\ \dot{p} &= -\nabla_x H = \nabla_x F(x) - A^T p - \nabla_x f^T(x)p & x(T) &= 0 \\ & & t &\varepsilon [0,T] \end{aligned} \tag{2.5}$$

Consider next the following definitions:

Definition 1. Admissible Control Set: The control set U is said to be admissible if

$$U = \left\{ u \,\middle|\, u \,\varepsilon\, \Omega \,;\, u, \dot{u} = \text{continuous, all } t \,\varepsilon\, [0,T] \right\}$$

Definition 2. Reachable Terminal Point Set: The terminal point set K is said to be reachable if

$$K(x_o,T) = \left\{ x(T) \,\middle|\, x(0) = x_o;\, u \,\varepsilon\, U, \text{ all } t \,\varepsilon\, [0,T] \right\}$$

From these definitions the following assumptions are imposed:

Assumption 1. Initial Point: The initial point x_o is assumed to belong to the set I, where

$$I = \left\{ x_o \middle| x(T) = 0 \in K(x_o, T) \right\}$$

Assumption 2. Unrestrained u: Assume that

$$\max_{u \in U} H \text{ exists}, \qquad \text{all } x, p \in E^n$$

and that the maximizing u is an interior point of an arbitrarily large range Ω.

Assumption 1 assures that the required terminal state is reachable in the allotted time T with an admissible control. Assumption 2 insures the smoothness of the control u for the least squares fitting, consistent with the definition of U. Neither assumption is regarded as especially restrictive. Note that the limiting effect of the u^2 term in the cost index makes the second assumption a virtual certainty.

Given the above definitions, relations, and assumptions, the open-loop trajectories x^*, u^* can now be defined from Pontryagins well known necessary conditions (3) for minimizing the cost index J subject to the plant constraint. Thus, x^*, u^* are to be selected as follows:

Open-Loop Trajectories: The trajectories $x^*(t)$, $u^*(t)$ are taken to be those open-loop time responses determined by the following conditions:

1. Initial point $x_o \in I$.

2. The differential system (2.5).

3. The control $u^* \varepsilon U$ determined such that Assumption 2 holds, and

$$H(p,x,u^*) \geq H(p,x,u)$$
$$u \varepsilon U$$

4.[1] If T = free, then $H(p,x^*,u^*) = 0$.

The adjustable cost functional provides selectivity to the designer in formulating candidate responses for the solution to his problem. All open-loop responses defined above are potentially useful candidates. Minimizing solutions imply an added property and are then said to be optimal in the sense of the following definition:

<u>Definition 3. Optimal Open-Loop Trajectories</u>: Open-loop trajectories are said to be optimal if

$$J(x^*,u^*,T) \leq J(x,u,T)$$
$$u \varepsilon U$$
$$x_o \varepsilon I$$

In most of the applications of the quadratic index, an optimal solution is an expressed goal. In the next section, on the selection of Q, it is shown how optimality is only incidental to the application of this study. However, because of the traditional interest in preserving optimality and for comparative purposes with the point of view in this

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[1]Since H turns out to be constant, the boundary condition 4 can be checked at any single point in time on the interval. Alternatively, the condition is deleted completely when T is prespecified.

study it is relevant to consider the property further here. The remaining steps of this section therefore examine conditions sufficient to establish that an open-loop response is also an optimal response. In the development it is convenient to invoke the notion of convexity. Thus, consider the following definition:

> <u>Definition 4. Convex Function:</u> A differentiable scalar function $\Theta(x)$ is said to be convex on a set C if C is convex[1] and if
>
> $$\Theta(x) - \Theta(\bar{x}) \geq (x - \bar{x})^T \nabla_x \Theta(\bar{x})$$

Conditions more suitable than the definition as tests are given later.

The convexity concept allows statement of the following theorem on sufficient conditions, contained by the recent results of Mangasarian (52).

> <u>Theorem. Sufficient Conditions (Fixed Time T).</u> Given a fixed time (T) response x^*, u^* satisfying conditions (1,2,3) of the open-loop (necessary condition) trajectories, and subject to the added conditions
>
> 1. $F(x)$ = differentiable and convex.
> 2. $-p^T f(x)$ = differentiable and convex in x.
>
> Then x^*, u^* is an optimal response.

Note first that for the type 1 plant $f(x) \equiv 0$, so that condition 2 of the theorem can then be deleted. Secondly, note that in no cases are

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[1]The choice $C = \left\{x \mid x \in E^n\right\}$ is the one of interest here.

further restrictions placed on the form of $G(u)$ and $g(u)$. Proof of the theorem, as carried out in Ref. (52), depends on convexity (Definition 4) rather than the progressively weaker properties of pseudo and quasi-convexity defined in the literature (e.g. Ref. (53)).

By the definition and theorem on optimality, the following corollary for the free time case is evident.[1]

> <u>Corollary. Sufficient Conditions (Free Time T)</u>: Given the plant and cost constraints satisfying the following conditions:
>
> 1. Sufficient conditions of the theorem for all fixed $T \in (0, \infty)$
> 2. Condition 4 of the open-loop trajectories with some free time $T_f \in (0, \infty)$
>
> Then the free time response x^*, u^*, T_f is optimal, and gives the lowest possible cost figure J.

The theorem holds for differentiable functions F and f. If F and f are twice differentiable, as is the case under study, then a more easily applied convexity test, rather than the direct use of Definition 4, is available. Thus, a necessary and sufficient condition for convexity of the twice differentiable function $\theta(x)$ in an open convex set is positive semidefiniteness of the matrix D,[2] i.e.,

$$D = \left[\frac{\partial^2\theta(x)}{\partial x_i \partial x_j}\right] = \text{positive semidefinite.} \qquad (2.6)$$

---

[1]Ref. (7), pp. 497.

[2]Ref. (55), Theorem 3-1, pp. 100.

The test (2.6) is now applied to the conditions of the sufficiency theorem and the functions of the problem under study. Consider condition 1 of the theorem, convexity of $x^T Qx$. The test (2.6) reduces the condition to

$$Q = \text{positive semidefinite.} \tag{2.7}$$

from (2.6) the second condition of the theorem, needed for the type two plant only, will be met if and only if

$$D(x,p) = [-\nabla_x(\nabla_x f^T(x)p)^T] = \text{positive semidefinite.} \tag{2.8}$$
$$\text{all } t \in [0,T]$$

In the next section a point of view is given which shows how optimality of J can be, and perhaps most often is, incidental to the engineering problem.

Selection of the Parameters Q and T

Each design problem has a set of performance requirements and design constraints, including the plant. Thus, consider the performance requirement set P for the typical problem, where

$$P = \left\{ x(t), u(t) \;\middle|\; \begin{array}{l} x, u \text{ satisfy given performance} \\ \text{requirements, subject to design} \\ \text{constraints, including a plant.} \end{array} \right\} \tag{2.9}$$

In the open-loop phase of the design procedure, the designer seeks to adjust the parameters Q and T such that

$$(Q,T) \in S_p \tag{2.10}$$

where S_p is the subset of open-loop solutions which lie in P, i.e.,

$$S_p = \left\{ (Q,T) \;\middle|\; (x^*,u^*) \in P, \text{ all } t \in [0,T] \right\} \tag{2.11}$$

As with any design problem and procedure, both P and S_p are tacitly assumed nonempty. In suitably restricted cases, the validity of the assumption is known in advance. In engineering practice, however, where requirements may vary widely with the application, the comfort of this knowledge may sometimes realistically emerge after the fact.

The weights Q influence the response amplitudes, while T controls the time interval. The expectation is that with proper selection, the combined effect should provide, in a qualitative sense, a worthwhile range of candidate solutions, consistent with the plant constraint. Examples in the case studies of Chapter IV, along with other results in the literature,[1] lend confidence that this expectation will often be realized in practice.

Although the study is limited to constant weights Q, the use of time varying weights offers a reserve capacity for expanding the size of the solution set S_p. In such case, the methods of solution employed in the implementation carry over directly.

Further consideration in the selection of Q and T are given individually below.

(i) Selection of Amplitude Weights Q

A number of writers have discussed methods of weight selection for the quadratic index. Fisher (13) uses successive estimations from

---

[1] e.g., Refs. (1,5,13,20).

lower order dynamic approximations, and gives an example for a ninth order process. He indicates (through the comments of a reviewer) that further uses of this approach may be found in Refs. (37, 38). Merriam[1] discusses a philosophy of weight selection based on making the maximum allowable response errors contribute equally to the integrand of the cost index. In so doing he develops expressions for time varying weights.

In the present study, primary emphasis is given to the method discussed by Bell (1). The fundamental basis for the method is that stationary linear system responses, with arbitrary left half plane (i.e., stable) root locations, can be formed from the quadratic index with a diagonal choice Q, $T \rightarrow \infty$, and a stationary linear plant.[2]

Note first that under these conditions, a priori assurance exists that both the performance and solution sets P and S_p are non-empty. In fact, the solution $(Q, T \rightarrow \infty) \varepsilon S_p$ turns out to be unique. Secondly, because of widespread useage, the linear class of problem is especially important. Thus, a strong intuitive "feel" often exists among designers for specifying dynamic responses in terms of linear system criteria. The concept has special appeal since "to first order," the relative simplicity of linear analysis can be applied to nonlinear systems.

As discussed by Bell, these points can be exploited in the weight selection problem. Thus, Bell presents a convenient table[3] for finding

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[1]Ref. (5), pp. 186.

[2]As developed, plants with transfer function zeroes are omitted.

[3]Ref. (1), pp. 27, Table 2.

the inverse relationship of $(Q,T \rightarrow \infty)$ as a function of root locations and linear plant parameters. He then suggests that the table can serve as a useful guide in selecting Q for:

1. Other finite response times T.
2. For nonlinear plants, through use of an appropriate plant linearization.

Examples were given by Bell to support this conjecture. The list of applications is extended here in the case studies of Chapter IV.

It is of interest to note that the ultimate weight selection criterion $(Q,T) \in S_p$ (2.10) makes no explicit reference to optimality, in the sense of Definition 3. Moreover, a straightforward application of Bell's approach, using standard root location criteria, appears more often than not to give Q selections which are not positive semidefinite (and thus fail to meet the hypothesis of the sufficiency theorem). Yet, in a qualitative sense, the corresponding responses often yield typically acceptable results.[1] Results in the case studies of Chapter IV give illustration to this point. A notable exception is found with free time and a nonlinear plant in the second case study. A similar conclusion regarding the successful use of nonpositive Q was noted by Rynaski and Whitbeck.[2]

It should be noted that Bell's table and diagonal Q matrix are based on the use of normalized phase variable coordinates. Thus, Bell considers the differential equation

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[1]For example, this is especially so with reference to the degree of relative stability.

[2]Ref. (20), pp. 61.

$$\frac{d^n\theta}{dt^n} + b_1 \frac{d^{n-1}\theta}{dt^{n-1}} + \ldots\ldots\ldots b_n\theta = Ku$$

and writes the corresponding quadratic cost as

$$J(u) = \frac{1}{2}\int_0^T (y^TCy + u^2)dt \tag{2.12}$$

C = a diagonal matrix.

$$y = [y_1, y_2, \ldots, y_n]^T = \frac{1}{K}[\theta, \frac{d\theta}{dt}, \ldots, \frac{d^n\theta}{dt^n}]^T$$

Normalized phase variables may not always be an appropriate choice for the state representation. If the plant is stationary and linear, then a matrix M can be found such that

$$y = Mx \tag{2.13}$$

where x is the selected state. Then to within a scalar multiple, the cost index (2.15) for Bell's table is equivalent to the more general form (2.1) if

$$Q = Q_c = \frac{1}{K^2} M^TCM \tag{2.14}$$

When the plant is nonlinear, M can be found from the same plant linearization applied in the use of the table. The matrix Q_c will remain symmetric, but includes off diagonal terms.

(ii) Selection of Response Interval T

The response interval T must be specified. Three basic choices are available: (1) "long" fixed T, (2) "short" fixed T, (3) unspecified

T. The structure of the controller may be altered by the choice, thus offering some degree of control over the feedback hardware.

In a qualitative sense, a long T response tends to reach a close neighborhood of the terminal state in advance of the termination time. In such case a linear stationary plant turns out to require a linear gain feedback law. Moreover, the gains approach constants as T is increased. Conversely, a short T response does not reach the neighborhood of the terminal state until the final moment. Relatively large control amplitudes are needed to achieve the fast response. A stationary linear plant calls for a nonstationary linear gain form of feedback hardware.

Condition 4 on the open-loop trajectories serves to determine T when the selection is left unspecified. The value will vary with the initial condition x_o. A stationary linear plant requires a stationary nonlinear feedback law in the free time case.

Examples of all three response types are found in the case studies of Chapter IV.

Numerical Methods of Solution

The open-loop trajectories have been shown to require the solution of a two point boundary value problem. Adequate methods for the solution of this type of problem continue to be the subject of extensive research (39, 40, 41). In the present work two different approaches have been employed, the adjoint method and the flooding technique. Jazwinski (42) gives a detailed account of the former, while Kipiniak (26) gives numerous examples of the latter.

Some of the examples in the case studies of Chapter IV confirm the computational difficulties often found in the solution of two point boundary value optimization problems. Details of these difficulties are discussed in the examples as they arise. Both methods rely on extensive use of numerical integration.

A rather general numerical integration digital computer program, with options for initial and two point boundary value problems, was developed. The program was used exclusively for all the open-loop and active closed-loop control simulations of this work. The basic program is designed to accept the specifics of each problem (e.g., differential equations and input/output requirements) through the use of a set of five "plug-in" subroutines. Thus, for each new problem, added programming is limited to the subroutines, written to observe standard interface requirements with the bulk of the main processing routine. Data monitoring features were incorporated in the main program to automatically control and check the progress of the calculations, and to provide appropriate data dumps to aid in assessing convergence problems, matrix singularities, blunder errors, etc. The difficult nature of the open-loop optimal solutions was well illustrated by the continued frequency with which these diagnostics were employed.

A detailed description of the program and its use would require an extensive write-up, and is therefore not presented in this publication. However, because of the important role of the program in generating the results presented, a listing of the basic program and a sample set of subroutines is given in Appendix D. The basic features of the methods used in generating the open-loop solutions are briefly indicated below. Special characteristics are listed, but all well documented techniques are pointedly left to the references.

(i) Numerical Integration

Two numerical integration options were employed. A Modified Euler[1] method with truncation errors of the order h^3 was used where h is the integration step size. As implemented, the method requires two substitutions in the differential equations per step. Hence, the method has a distinct advantage in calculation time over the second option.

The second integration option was a vector adaptation of the fourth order Simpsons rule variation of the Runge-Kutta[2] method. The advantage of this option is its greater accuracy over the Modified Euler. Truncation errors are of the order h^5. The greater accuracy is achieved at the expense of calculation speed (roughly 2:1) since four substitutions in the differential equations are required per step.

A variable step size error control option was also included with the basic Runge-Kutta. This option was extensively exercised by some of the sharply breaking nonlinearities included in the Case Two Studies of Chapter IV. The error detector was a vector adaptation patterned largely after the Collatz approach, as discussed by Merriam.[3]

(ii) Adjoint Method

The adjoint method was used for the majority of two point boundary solutions in the first and third case studies (Chapter IV). The method

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[1] Ref. 43, pp. 93.

[2] Ref. 43, pp. 103.

[3] Ref. 5, Section F-2, pp. 355.

is iterative. Its development starts by writing linear perturbation equations from the actual 2n differential equations. The adjoint of these equations is then programmed, making a total of 4n differential equations.

The main objective of the method on each iteration is to construct a matrix S such that

$$S\,\delta p(0) = -Cx(T) \tag{2.15}$$

$$C = [c_{ij}] \qquad 0 < c_{ii} \leq 1$$
$$c_{ij} = 0 \quad i \neq j$$

where $\delta p(0)$ is a correction to the last trial of unknown initial conditions, $x(T)$ is the terminal state (error) on the last attempt, and C is a diagonal weighting matrix used to automatically control the convergence step size. Thus, the matrix S is equivalent to a matrix of partial derivatives relating linear perturbations in the terminal state to perturbations in the unknown initial conditions. Note that the partial derivatives "span" the integration interval T. In the adjoint method the partials are derived numerically by a sequence of back integrations of the adjoint equations, with appropriately selected initial conditions. The key advantage is that the derivatives are computed by the relatively accurate process of integration, thus avoiding the possibility of less accurate attempts at numerical differentiation. Complete details of the very general formulation used in the program are given in Ref. 42.

Numerical inversion of S is by a Gauss-Jordan elimination.[1]

[1]Ref. 44, Chapter 1.

If the problem is completely linear, the matrix S gives a boundary value solution in one iteration. By definition, the matrix can be viewed as an error sensitivity matrix. The extreme co-state sensitivities often found in control system optimizations are well illustrated by some of the examples in the third case study (Chapter IV).

(iii) Flooding Technique

In the flooding technique, the two point boundary value problem is solved as an initial value problem. Note that only typical open-loop trajectories are required. Thus, repeated back integrations from the desired terminal state, with arbitrary selections for the unspecified end conditions, gives a flood of trajectories into the state space. If any or all of these cover the desired space then the open-loop problem is solved.

If the response interval T is unspecified, selection of the unknown conditions is not arbitrary, but must satisfy condition 4 on the open-loop trajectories, i.e., the co-state vector p must be selected such that

$$p^T(f(0) + g(u)) - u^2 = 0 \qquad (2.16)$$

The flooding technique was used exclusively for all the applications in the second case study (Chapter IV). Contrary to the adjoint method, flooding was reasonably easy to carry out in the latter case. However, in problems of larger dimension, or where the co-state sensitivities are exceptionally high, the flooding approach may be difficult to apply. This situation is demonstrated in the third case study. In some cases a combination of the adjoint and flooding techniques may be useful. Thus the adjoint method may be used on a limited number of trajectories, to establish appropriate regions for the co-state selections. Then flooding can be used to fill out desired regions of the state space.

Note that flooding would be considerably more difficult (though yet possible) if the plant equations (2.2) were not autonomous. If such were the case, <u>sets</u> of flooding solutions would be required to account for the nonautonomous components. Similar remarks apply, in fact, regardless of the method of solution

CHAPTER III

CLOSED-LOOP CONTROL SURFACE APPROXIMATION BY LINEAR LEAST SQUARES

Linear Least Squares Analysis

Chapter II has covered essential points in the first two steps of the design method (Chapter I). Upon completion of these steps an accumulation of discrete open-loop data points $(u^*(t_k),\ x^*(t_k))$ are available. For convenience these are stored on magnetic tape by the computer as the open-loop solutions are generated. This chapter presents the essential points of the remaining three steps of the design method. The purpose of the latter are to construct and test a closed-loop feedback law satisfying the original design requirements. As discussed in Chapter I, the basis for this construction is the fitting of selected forms to part or all of the open-loop data surface using linear least squares. The selected data on each construction is often referred to as the <u>data base</u>.

Consider some further definitions. Equation (1.2) is now stated more precisely as

> <u>Definition 5: admissible feedback control law.</u> A feedback control law is said to be admissible when it is taken from the class of functions

$$\hat{u} = k^T Z(x,t) \tag{3.1}$$

where

u = feedback control signal
k = an r vector of arbitrary constants
Z = a vector of r linearly independent continuous functions, with continuous first derivatives in the arguments x and t.

Note that the admissible control surface (3.1) is linear in the gain vector k. Furthermore, the definition can be extended to any auxiliary function reducible to this form.[1] Selection of a given form corresponds to Step 3 of the design method (Chapter I).[2] Selected functions will often be referred to as basis functions.

Definition 6: Least Squares Control Law. A feedback law is said to be a least squares control law when the gain vector k is determined such that it satisfies the least squares criterion

$$E(k) = \sum_{j}^{M} [u_j(k) - u_j^*]^2 \qquad (3.2)$$

$$\min. \ k$$

where

u_j^* = the value of the open-loop control at the jth point of the data base set.

u_j = the value of the control law (3.1) computed from the open-loop state and time x^*, t values, taken from the jth point of the data base set.

It is well known that a solution to the linear least squares problem does

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[1] For example, by the use of logarithms to reduce the functions $u = Z_1^{k_1} Z_2^{k_2} \ldots Z_r^{k_r}$ or $u = ce^{k_1 Z_1} e^{k_2 Z_2} \ldots e^{k_r Z_r}$.

[2] For a discussion on efficient means of selection of the functions Z_i, see Appendix E.

exist and is unique.[1] The minimization is a parameter optimization problem in r variables and is formed by setting the gradient of the scalar E(k) with respect to the gains to zero. Using (3.1) this gives the vector algebraic equation

$$\nabla_k E(k) = \nabla_k \left\{ \sum_j^M [\hat{k}^T Z_j - u_j^*]^2 \right\}$$

$$= 2\left\{ \sum_j^M Z_j Z_j^T \hat{k} - \sum_j^M u_j^* Z_j \right\} = 0$$

Since the relation is linear in k the least squares gain solution is such that

$$\hat{k} = B^{-1} a \tag{3.3}$$

$$A = \left[\sum_j^M u_j^* Z_j \right]_{r \times 1} \quad B = \left[\sum_j^M Z_j Z_j^T \right]_{r \times r}$$

Note that B is rxr in dimension, regardless of the size (M) of the data base. Most important, the construction uses open-loop solutions in the form of tabular data. Thus, explicit closed form solutions are not required.

It is remarked that the control (3.1) may be forced to include a constant term by setting one of the Z vector elements at unity. A similar effect is incurred by performing the fitting with respect to mean adjusted variables, i.e., $\underline{u} = u - u_o$, $\underline{Z} = Z - Z_o$, where u_o, Z_o are the arithmetic

[1]see e.g., Ref. 48, pp. 181.

mean of the values in the data base. This procedure is employed on the functions of the first case study of Chapter IV.

A computer program for the Burroughs B-5500 which includes an implementation of the above computations was obtained from the Rich Electronic Computer Center at Georgia Tech. This program was used exclusively for all the least squares calculations of this work, and is thoroughly described in Ref. 36.

Goodness of Fit: Least Squares

If the number of data points equals the number of terms in the control function (i.e., $M = r$ in (3.1), (3.2)) then a perfect fit will be found for virtually any set of r elements $Z(x,t)$ selected for the basis function. This is the result of r unknowns satisfying r equations, and should generally give poor interpolations and extrapolations at points not on the data base. Thus, to give some assurance of an adequate fit over the entire control surface, the calculation is normally based on a large statistical sampling, i.e., $M \gg r$.

The value of the sum of squares of error (3.2) is one obvious criterion for assessing the quality of the least squares fit. A very commonly used alternate, the square of the multiple correlation coefficient (R^2) can be computed from the formula

$$R^2 = \frac{\left(\sum^{M} \hat{u}_i u_i^*\right)^2}{\left(\sum^{M} \hat{u}_i^2\right)\left(\sum^{M} u_i^{*2}\right)} \tag{3.4}$$

The correlation R has a geometrical interpretation, namely, the cosine of the angle between the vector of estimation elements $\hat{u}_i$ and the vector of data elements u_i^* in the open-loop data base. Thus, a value of zero would indicate no correlation, while $R^2 = 1$ signifies that all the observed data u_i^* lies exactly in the estimation hyperplane.

Goodness of Fit: Performance

Under proper conditions, the Gauss-Markov theorem[1] states that a least squares estimate is optimal (i.e., smallest variance). The conditions of the theorem are that the data u_i^* belong to the assumed class of estimation functions $\hat{u}$ but include uncorrelated random deviations with a common variance and zero mean. These conditions do not generally hold in the present application. Furthermore, true "optimality" is implied by the design requirement of each problem, or at least by the cost index J, as discussed in Chapter I. Thus, goodness of fit in the least squares sense, though convenient, does not imply a well fitting solution in the sense of performance.

Because of this disparity in fitting criterion, all candidate gain solutions $\hat{k}$ are tested for adequate performance by active simulation on the digital computer. This is the final step (Step 5, Chapter I) of the design method. The need for this step is amply demonstrated by the examples of the three case studies of the next chapter. It will be shown that the choice of basis function and selection of the data base can be key variables in the success of the method.

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[1]Ref. 47.

CHAPTER IV

EMPIRICAL CASE STUDIES

Empirical Analysis Formulation

The previous chapters have discussed aspects associated with the steps of the design method. Before presenting the empirical applications of this chapter it is appropriate to define the empirical variables and identify the manner in which the results are expressed. As a means of indicating the types of variables included in the analysis, it may be helpful to view these basic items as vectors, having both quantitative and qualitative elements, in an "empirical analysis space." This descriptive device is illustrated in Figure 3, where a performance vector is portrayed along with the five basic considerations of the empirical analysis which serve as the independent variable groupings.

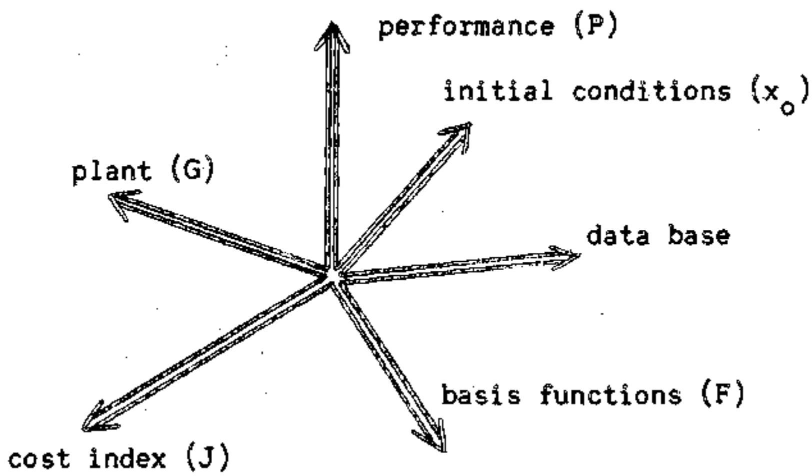

Figure 3. A Vector Representation of Analysis Space.

The summation of all possible applied combinations define a multidimensional empirical surface

$$P = \xi(G,J,F,D,x_o)$$

which provides a certain abstracted description of the design method. Thus, the objective of the empirical analysis is to generate sufficient points on this surface such that a worthwhile description of the design method can be reached, consistent with the necessary costs incurred. While each of the items are of fundamental importance, the size of the multidimensional "subspaces" in which they reside requires that some restrictions be placed on the data accumulation and its presentation. However, a significant understanding of the problem can be gained by selective empirical testing and it is hoped that some worthwhile balance has been achieved in the results presented.

The quantitative and qualitative manner in which each of the six basic empirical variables identified in Figure 3 are defined is now discussed. As an aid to the reader, each of the specific sub-items to be found in each of the case studies are then summarized in Table I.

1. <u>Performance</u>

   a) trajectory plots

   b) J_T: cost index at fixed time T[1]

   c) ε_T: terminal error at fixed time T

   d) J_ε: cost index at fixed error (free time)[1]

   e) T_ε: time at fixed error (free time)

   f) t_r: rise time for fixed initial condition

   g) t_s: settling time for fixed initial condition

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[1] In these comparisons, J must be convex, i.e., Q = positive semidefinite.

The use of phase plane and time history trajectory plots (a), even though qualitative in nature, presents a valid means of comparing results. Quantitative performance measures for nonlinear systems appear notably absent in the literature.[1] The scalar cost and Euclidian norm of error (b,c) can be of interest when compared at the specified fixed time T. In problems where the time T is unspecified, however, it is misleading to make these comparisons at a prespecified time. Thus, at a given time the cost (J) of a suboptimal trajectory is frequently lower than the value associated with the true optimal path with unspecified T, although this occurs only at the expense of having moved a shorter distance toward the origin. By definition, at the origin the nonoptimal case will ultimately accumulate a greater total cost J, and generally will take a longer time to do so. Hence, for the important cases where T has been left unspecified, items (b,c) are replaced by cost and time required to reach a given measure of error (d,e). As a matter of convenience, this error measure is taken as the point when a trajectory reaches, and subsequently remains whithin, some small preselected hypercube about the origin. The rise and settling time measures (f,g) are defined similar to their well known invariant counterparts from linear system analysis, with one exception. Since the considered cases are regulator problems, amplitudes are measured in terms of initial displacements rather than step inputs. Thus, rise time is obtained by starting on the axis of the most fundamental state variable (e.g., output position, with all other state variables therefore at zero) and measuring the time interval to proceed

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[1] This presents a distinct difficulty. Actual performance measures, defined by the designer, will vary with the application. Thus, any measures selected here are useful only to the extent that the designer can extrapolate the results of the study to the individual needs of his problem.

from 90% to 10% of the initial displacement. Settling time is the time interval to reach and remain within ±5% of the original displacement magnitude. Unlike the case of stationary linear systems, the rise and settling times for nonlinear systems are initial condition dependent (as is true for all the items a-g), and must therefore be presented as such to have valid meaning. Nevertheless, nonlinear comparisons of this type are readily computed and hence these familiar response time measures are utilized here. The fact that measures other than the cost index J , (the sole basis for defining the optimal trajectories), are included for assessing the results gives recognition to the less than complete role of the quadratic index, even for conditions where optimality is assured.

2. <u>Plant</u>

a) order, form, etc. of linear dynamics

b) shape, extent of nonlinearity

One of the independent variable groups is that of various possible plant combinations. Changes at the plant level represent the highest order of change in the five group hierarchy of variations (Figure 3), since to a large extent changes at this level call for a repetition of most of the other basic variations to properly assess the new plant model. Accordingly, this level of variation is necessarily more restricted than the others. As indicated, plant variations are subgrouped into differences in the linear dynamics and differences in the nonlinearity. Three empirical cases are analyzed. The first of these consists of some low order plant dynamics in cascade with a linear gain. As such, a notable feature of this problem is that the true feedback control law can be found in closed form, thus permitting the design method to first be tested in full light of the true solution. By considering the fixed response interval T at both "long" and "short" values, a considerable change occurs in the structure of the open-loop control function. Hence the first case is a study of two related problems. The second case is a sequence of four problems, consisting of the same form of low order plant dynamics in cascade with a nonlinear element whose shape is varied in turn from the linear case to successively greater degrees of nonlinearity. The third study is most notable for its consideration of the case where higher order plant dynamics are involved. In this study a fourth order, type 2 servo with nonlinear, two-phase control motors is considered. This problem is useful in emphasizing limitations of the design procedure.

3. Cost index

a) response time interval (T)

b) weighting coefficients (Q)

Three types of response intervals are considered, short fixed time ($T \ll t_s$), long fixed time ($T \gg t_s$) and unspecified (free) time. As can sometimes be shown analytically, the choice of these response interval types may greatly effect the structure of the optimal control. Thus, attention is given to this aspect in the analysis. Since application of the design method requires a proper selection of the weights (Q), this further cost index related aspect is given careful consideration. In the second and third case studies, particular attention is given to the weight selection method suggested by Bell (1).

4. Basis functions

a) assorted control functions deemed appropriate to the application (see Table 1).

The success of the least squares fitting depends basically upon the capability of the assumed control law to duplicate the open-loop control. This item is of relatively low order in the hierarchy of empirical group variations. Consequently, each problem is considered under various conditions with a number of assorted basis functions

5. <u>Data base</u>

a) number of data points

b) size of data distribution

c) data weighting

It will be shown that the proper selection of the data base is often an important consideration in the success of the least squares fitting. Size (b) refers to the size of the enclosed region in state space where the data is taken from. Data weighting (c) is achieved by concentrating data selections in regions where the heavier weighting is desired, e.g., by crowding more points near the origin, or in some cases near locations where the signal is known to cross zero.

6. <u>Initial conditions</u>

a) inside data base

b) outside data base

Testing of the designs under varying initial conditions is an essential step in the design method (see Design Method Steps, Chapter I), since it directly tests the ultimate objective of the design method, i.e., the ability to properly relate the control signal as an explicit function of the state. As such it is the lowest order empirical variation in the five group hierarchy of basic considerations in Figure 3. This level is invoked by making a set of test runs for each combination of the higher

Table 1. Summary of Empirical Studies

Other Variations		Plant Variations					
		Case 1	Case 2				Case 3
			N=2	N=3	N=10	N=50	
Performance	trajectory plots	x	x	x	x	x	x
	J_T:fixed time cost						x
	ε_T:fixed time error	x					
	J_ε:free time cost		x	x	x	x	
	T_ε:free time for error		x	x	x	x	
	t_r:rise time		x	x	x	x	
	t_s:settling time		x	x	x	x	
Cost Index	short fixed T	x					x
	long fixed T	x					x
	free T		x	x	x	x	
	Q variations					x	x
Basis Functions	L: $\hat{u} = -k^T x$	x	x	x	x	x	x
	S: $\hat{u} = -k^T x - x^T M x$	x		x			
	Q: $\hat{u} = -(x^T k x)(x^T \delta)$				x	x	
	TVG: $\hat{u} = -\Phi(k,t_s)^T x$	x					x
Data Base	number of points	x					x
	size	x		x			
	weighting				x		
Initial Conditns.	inside data base	x	x	x	x	x	x
	outside data base	x		x			

order groups given previously. As indicated, the variations in initial conditions are subdivided into two catagories, those existing within the region of the data base (a) and those which are external to it (b). The latter case represents the extrapolation of the basis function outside of the region of fitting, and thus will tend to give successful results only when the basis function is a good representation of the true optimal surface. Conversely, it is demonstrated that many basis functions can be made to give adequate performance in sufficiently restricted regions of operation.

This completes the descriptions of the six basic dependent and independent groupings considered in the empirical studies, as displayed in Figure 3. The summary of Table 1 has been prepared as an aid to the reader in determining what multidimensional combinations have been tested and where these combinations may be found.

Case One

(i) Analysis

The first case study considers a second order plant with a linear gain function. The solution for the feedback control law can be obtained, though tediously, in closed form without resorting to the use of the considered design method. The purpose of selecting this problem for the first case study is to allow testing of the design method on a problem where everything about the true solution is completely known in advance. Thus, the problem affords a special insight not usually offered by other possible problems in the considered class. Furthermore, this experience carries over to the latter more typical cases of interest, while the closed form solutions do not.

Figure 4 shows a block diagram of the considered plant and indicates the form of the associated transfer function.

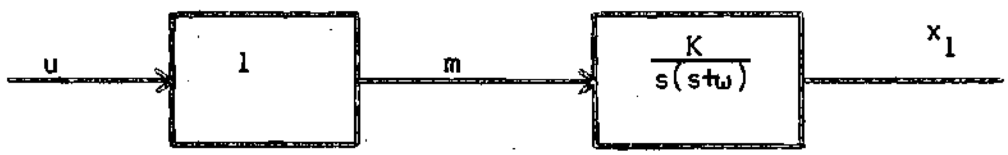

Figure 4. Plant Block Diagram

The relatively simple state, co-state, and adjoint differential equations which describe the plant and yield the open-loop, two-point boundary solutions are documented in Appendix A.

The cost index is evaluated for a fixed time T. Using the weight selection method of Bell as discussed here in Chapter II, the diagonal matrix

$$Q = \begin{bmatrix} .1 & 0 \\ 0 & -1 \end{bmatrix}$$

is chosen. The basis for this choice corresponds to a model set of closed-loop root locations which yield a damping ratio $\zeta = 0.707$ and an undamped natural frequency $\omega_o = 0.567$ rad/sec. The identical problem was treated by Bell[1] for a response interval of T=10 sec. Note the negative weighting coefficient in Q, which arises from a need to offset the inherent damping in the plant with sufficient positive rate feedback to achieve the oscillatory closed-loop response.[2]

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[1]Ref. 1, Chapter III.

[2]As a result, Q is not positive semidefinite.

Some typical displacement and control time histories are compared in Figure 5 for several fixed response intervals (T). The convergent nature of the responses can be observed as T is increased. In the succeeding analysis, response intervals of 2.5 and 15 seconds are considered, corresponding to "short" and "long" T. Typical projections of the open-loop data on the phase plane for the two cases are shown in Figures 6 and 7.

The significance of the long and short T cases for this study is best revealed by referring to their known closed form control functions. The very tedious algebra is reduced for the present case by Bell. The result is of the form

$$u^*(t_g) = -k_1(t_g)x_1 - k_2(t_g)x_2$$

$$t_g = T - t : \text{time-to-go}$$

and has the property that as $T \to \infty$

$$k_1(t_g) \to k_1$$

$$k_2(t_g) \to k_2$$

Thus, as T becomes large with respect to the closed-loop settling time this case approaches the simple constant feedback gain result

$$k_1 = 0.321$$

$$k_2 = -0.198$$

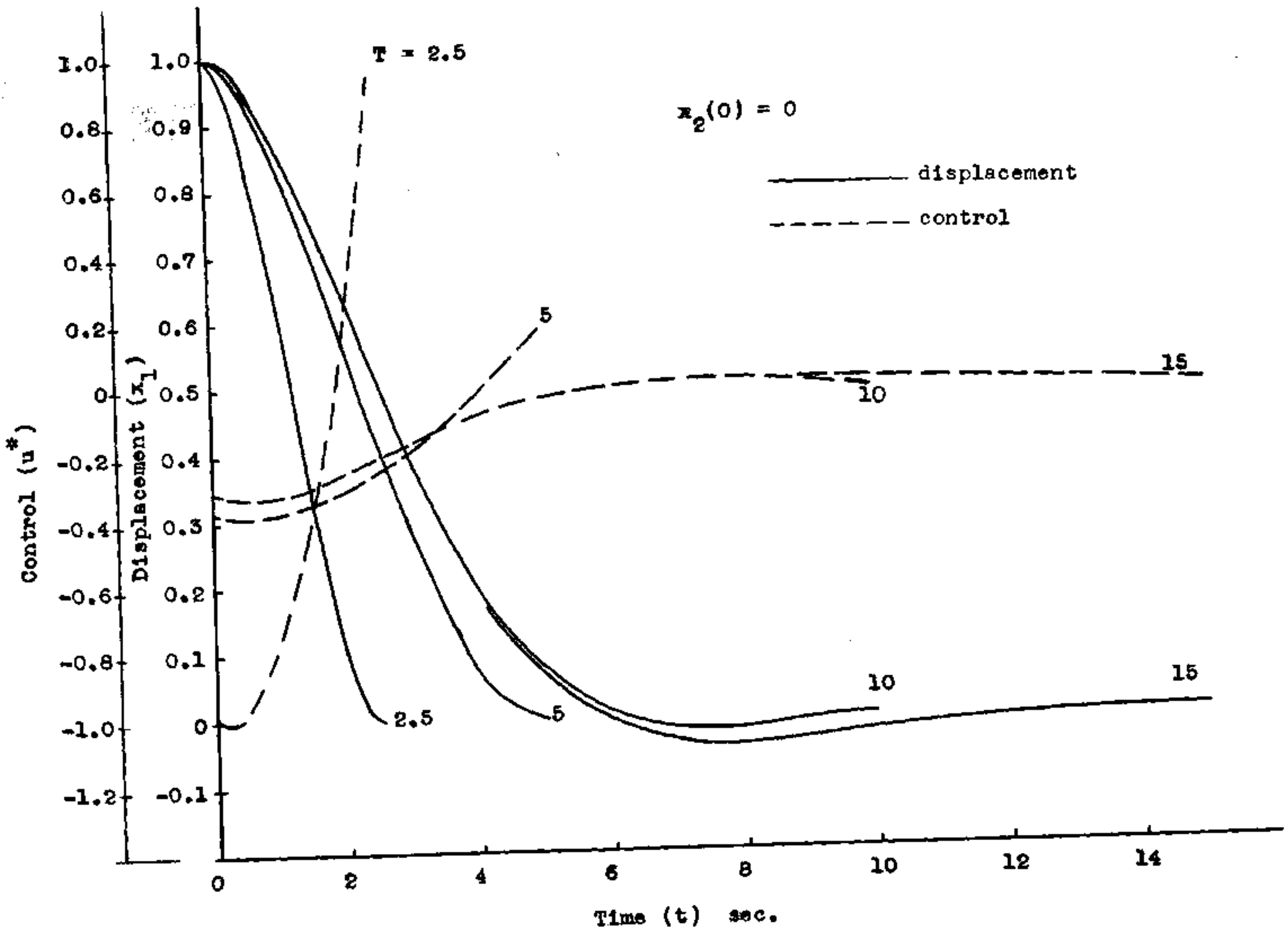

Figure 5. Typical Case One Time Histories.

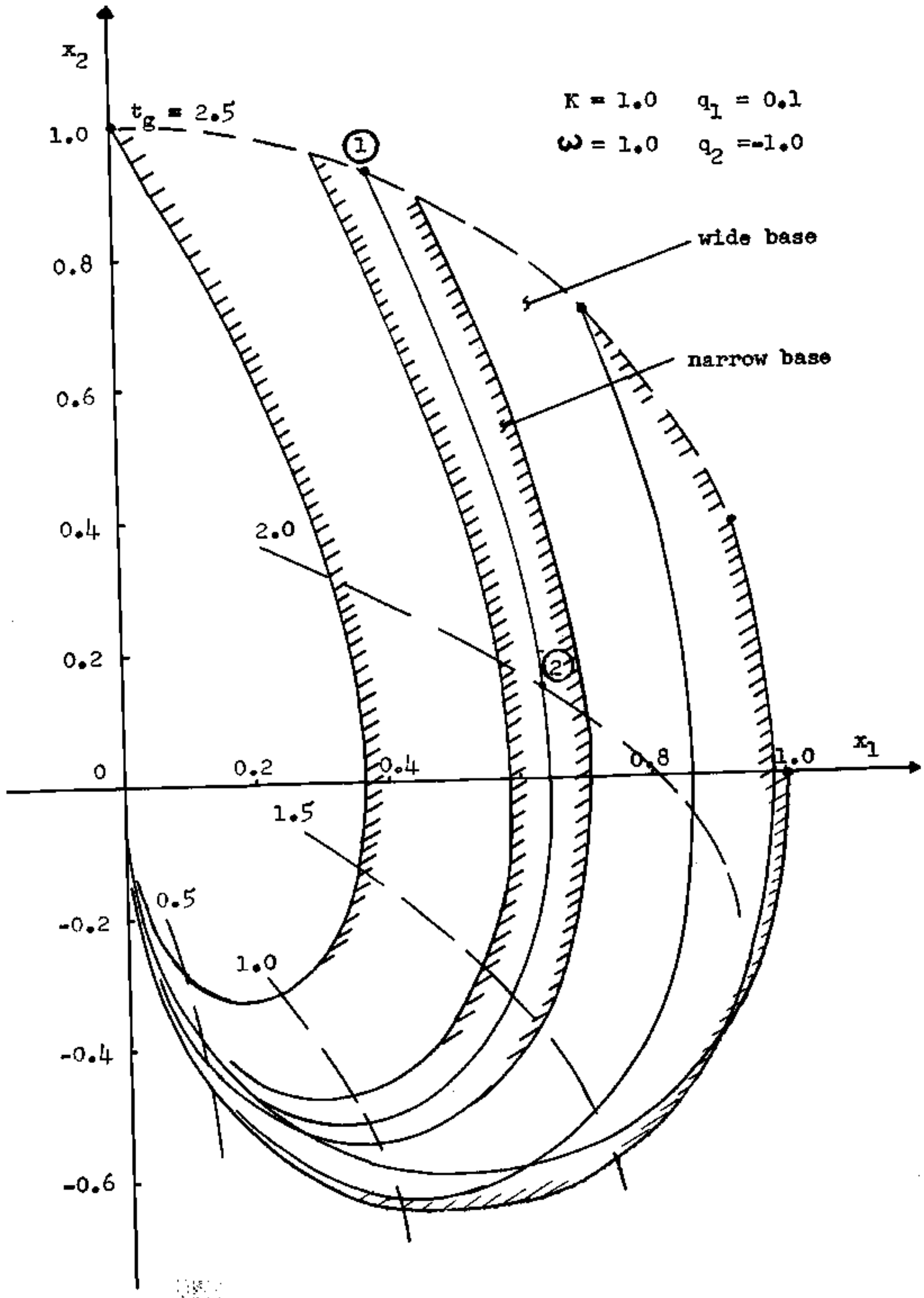

Figure 6. Case One Data Base (T = 2.5 sec.).

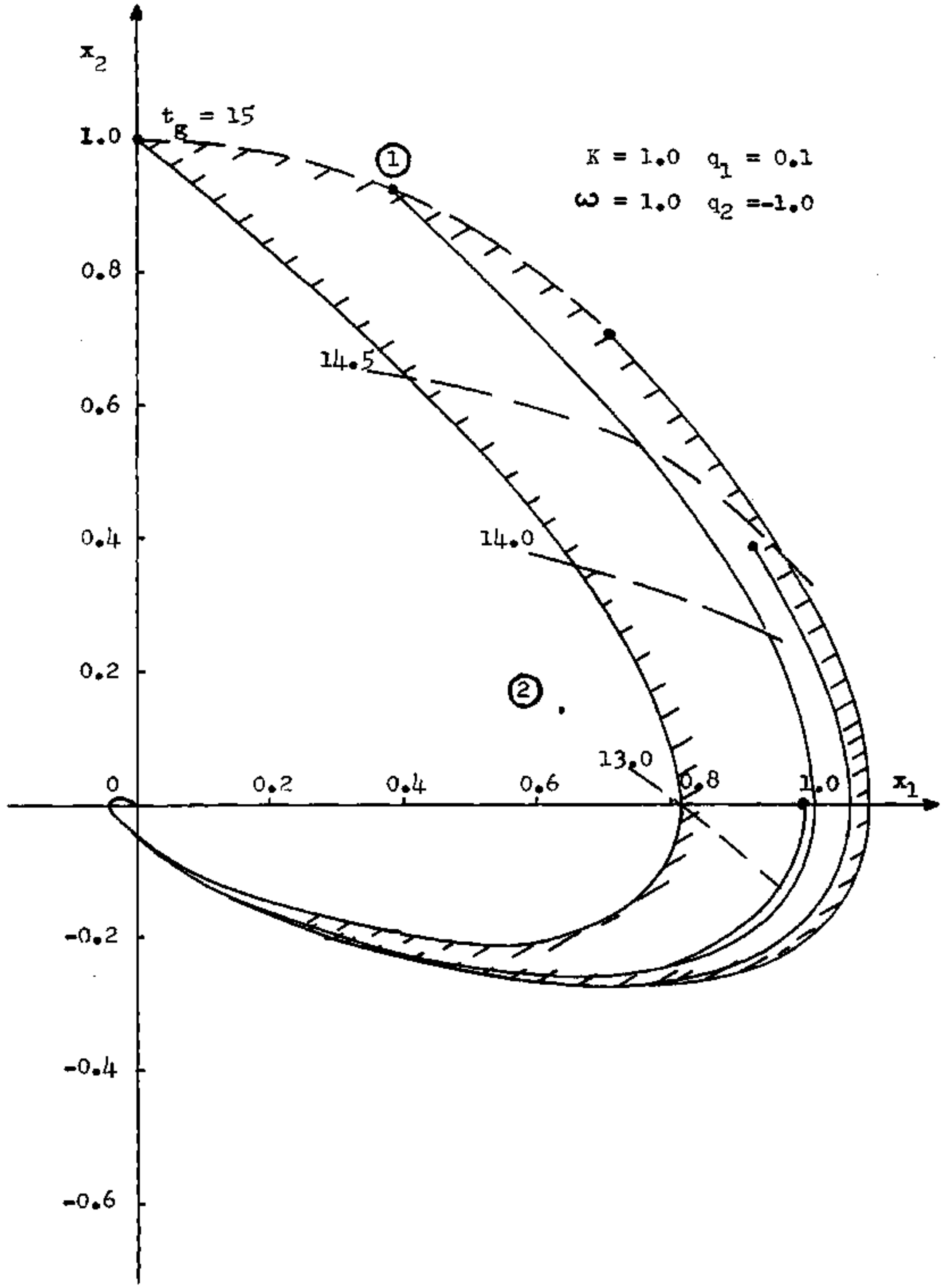

Figure 7. Case One Data Base (T = 15 sec.).

Thus, one would expect to find that application of the least squares procedure to a control law based on, say, a second order power series in the extended state variables (i.e., x_1, x_2, t_g) should meet with no more than moderate success in the short T problem. Conversely, one would hope the least squares analysis could produce a reasonably good controller with some form of time varying gains (TVG) model. A second test would be to see how effectively the least squares approach could lead to the simple constant linear gain result in the long T problem, and just how good the resulting approximation turned out to be. Quantitative answers to these and other questions are contained in the results.

Three specific forms of basis functions are analyzed in this problem. These are[1]

$$\hat{\underline{u}} = -k^T \underline{z} - \underline{z^T K z} \text{ : second order model} \tag{4.1}$$

$$z = \begin{bmatrix} x_1 \\ x_2 \\ t_g \end{bmatrix}, \quad k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}, \quad K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix}$$

$$\hat{\underline{u}} = -\underline{(K\tau)^T x} \text{ : 3rd order TVG model} \tag{4.2}$$

$$\tau = \begin{bmatrix} 1 \\ t_g \\ t_g^2 \\ t_g^3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix}$$

[1]The underbar signifies mean adjustment.

$$\hat{\underline{u}} = -(\underline{K_i \tau})^T x \quad : \text{ linear TVG model with time segmented gain switching} \tag{4.3}$$

$$\tau = \begin{bmatrix} 1 \\ \\ t_g \end{bmatrix}, \; K_i = \begin{bmatrix} k_{11} & k_{12} \\ \\ k_{21} & k_{22} \end{bmatrix}_i \quad \begin{array}{ll} i = 1 & (0 \leq t_g < t_1) \\ i = 2 & (t_1 \leq t_g < t_2) \\ i = 3 & (t_2 \leq t_g < T) \end{array}$$

$$t_1 = 0.6 \qquad T = 2.5$$

$$t_2 = 1.4$$

The switching times t_1, t_2 were selected in a somewhat arbitrary way which gave higher data point densities to the data base regions closest to the terminal end.

Other aspects considered are various data base and initial condition combinations for the long and short T cases, as summarized in Table 1. Specifically, the data base variations consist of the following size and number combinations:

Data Base:

- large no. : 286 data points
- reduced no: 130 data points
- wide size (WS): the entire regions enclosed by trajectories shown in Figure 6 and 7 respectively.
- narrow size (NS) (T=2.5 only): the narrower enclosed region marked in Figure 6

For convenience, a symbolic reference is made to these combinations in the plotted results. Thus WS-130 denotes a wide size, 130 point combination, NS-286 a narrow size, 286 point combination, etc.

The effect of initial condition variations are considered in terms of the following two standard cases:

$$\text{standard initial conditions} \begin{cases} (1) & \text{ideal} \\ (2) & \text{adverse} \end{cases}$$

The circled numbers are used for identification in the plotted results. The actual locations of these points in the phase plane are identified on Figures 6 and 7. In the short T problem (Figure 6), condition one is placed on the unit circle of the phase plane corresponding to a central trajectory common to both the wide and narrow data base sizes. As such it is "centered" in both data bases and hopefully represents an ideal starting point for the test simulations. Condition two is taken as the x_1,x_2 phase plane projection of the same trajectory, chosen at an intermediate time-to-go point (2.0 sec.).

Thus, since T is fixed at 2.5 seconds, the open-loop path starting from this point falls completely outside of the data base in extended state space, and therefore represents an adverse test of the control law. The same values are used in the long T problem, as shown in Figure 7.

As indicated in Table 1, performance comparisons for this problem are limited to an rms measure of terminal error at time T (zero time-to-go). Early experience with this problem indicated that the most difficult task for the control functions to attain was a reasonable terminal error in the allotted time T. Regarding the cost index J, it is qualitatively remarked that control functions which gave a low terminal error always resulted in very near the open-loop values of J.

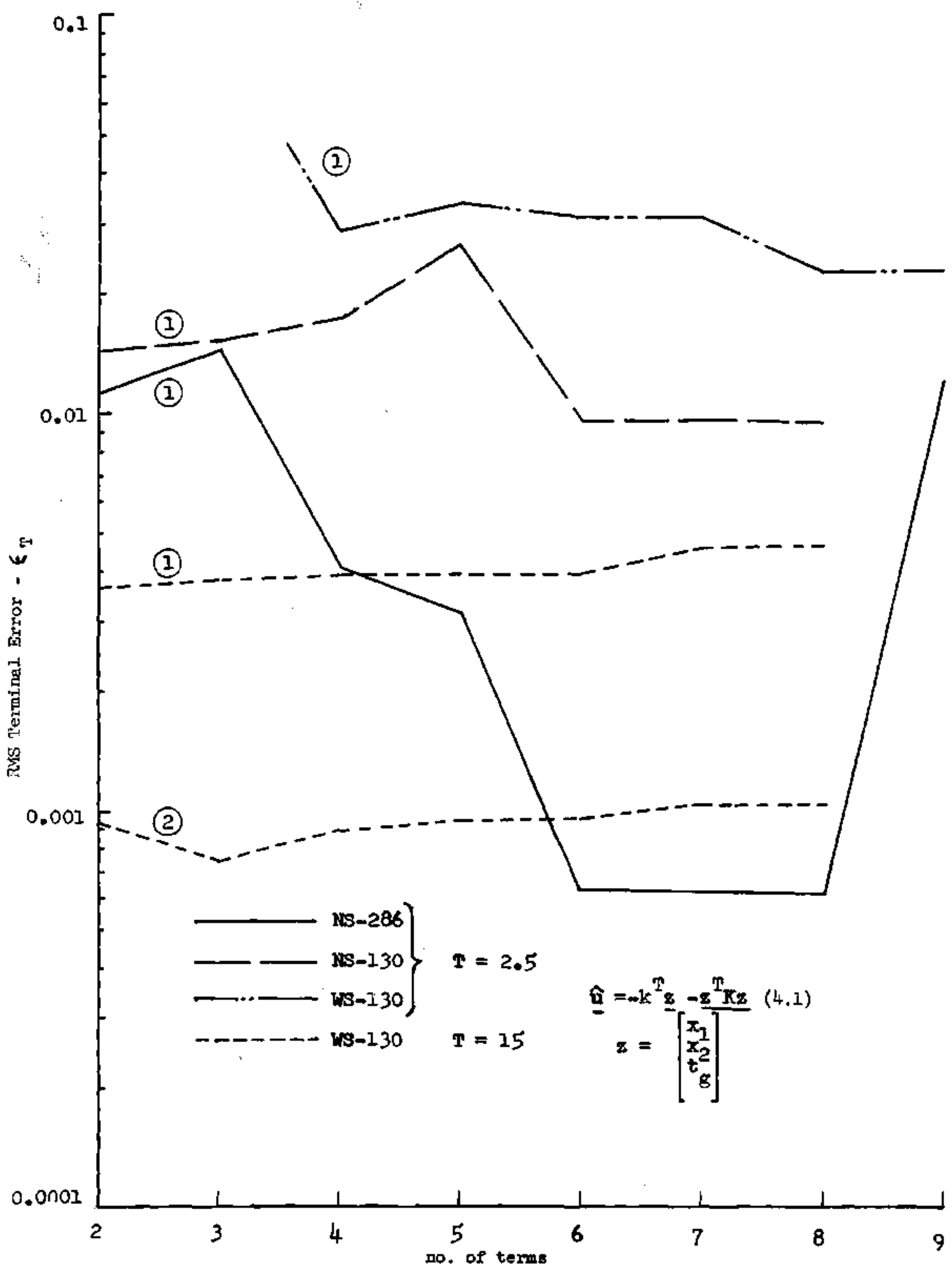

Figure 8. Case One Terminal Accuracies (Second order controls).

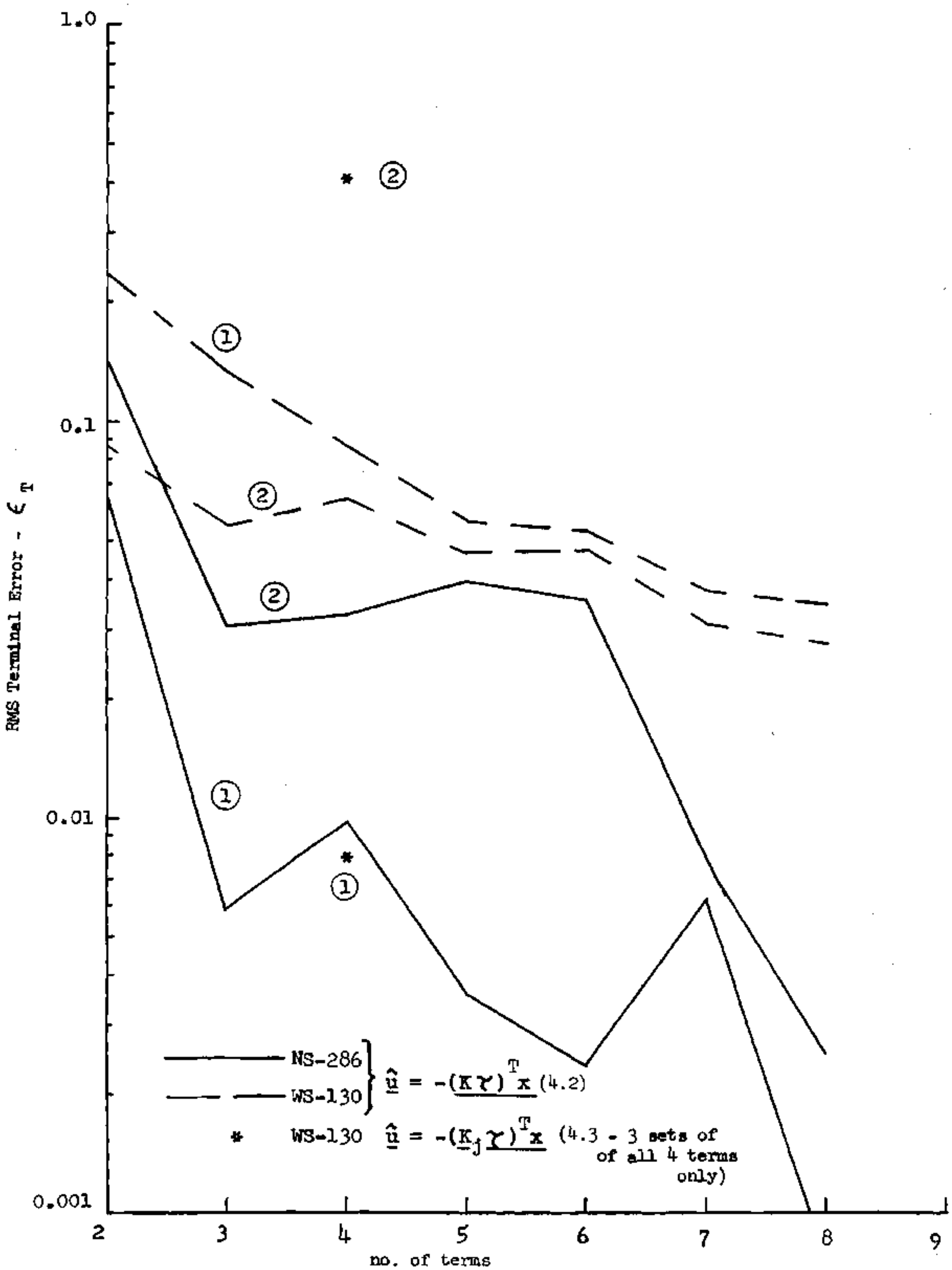

Figure 9. Case One Terminal Accuracies (TVG controls).

Table 2. Least Squares Summary (Case One, T = 2.5)

No. of Terms	Second Order (4.1) NS-130		Second Order (4.1) NS-130		Second Order (4.1) NS-286		TVG (4.2) NS-130		TVG (4.2) NS-286	
	R^2	added term[1]	R^2	added term[1]	R^2	added term[1]	R^2	added term[1]	R^2	added term[1]
2	.962392	t_g plus tg^2	.802495	x_1 plus $x_1 t_g$	.976733	x_1 plus x_1^2	.802495	x_1 plus $t_g^3 x_2$	.932165	x_1 plus $t_g^3 x_2$
3	.976548	x_1	.891898	t_g	.986080	t_g	.891898	$t_g^2 x_1$	.980661	x_2
4	.988613	$x_2 t_g$	.928739	$x_1 x_2$	.999403	x_2	.928739	$t_g^2 x_2$	.989046	$t_g^2 x_2$
5	.989164	x_1^2	.943062	x_1^2	.999426	t_g^2	.943062	$t_g^3 x_1$	.995941	$t_g^3 x_1$
6	.989477	x_2	.946988	t_g^2	.999726	$x_2 t_g$	.946988	$t_g x_2$	.996401	$t_g x_2$
7	.989499	$x_1 x_2$	.946979	x_2	.999730	x_2^2	.958873	$t_g x_1$	.998156	$t_g x_1$
8	.989814	x_2^2	.960692	x_2^2	.999733	$x_1 x_2$	.960692	x_2	.998771	$t_g^2 x_1$
9	.989837	$x_1 t_g$	--	--	.999738	$x_1 t_g$	--	--	--	--

[1]For k terms the control function consists of the added term plus all terms used in the k-1 term function. All terms mean adjusted.

The rms error comparisons of the more interesting results from the simulation runs are presented in Figures 8 and 9. These plots display the terminal error figures versus an increasing number of terms in the basis functions, up to the total number given for the functions (4.1) and (4.2). For the segmented TVG model (4.3), however, only the full four terms in each time segment are considered. For each fixed number of terms retained in the former two cases an attempt was made to reject those excess terms which contribute least to reducing the sum of squares of error.[1] Thus, this form of comparison gives an indication of changes in the assumed control function models. The selected terms of the second order and TVG control models are listed for each of the short T runs in Table 2. The corresponding least squares correlations (R^2) are also given.

(ii) Discussion

Consider first the results for the long T problem. Knowledge of the open-loop control function indicates a close approximation could result with a simple linear, constant feedback gain form. This form is available as a subclass of the second order model (4.1) with $K = 0$, $K_3 = 0$. Figure 8 shows that indeed, only two terms were needed to give relatively small errors at zero time-to-go. Furthermore the least squares analysis correctly identified these best two terms as the linear state feedback model, and gave the least squares approximation for the coefficients (with k_o the net constant) as

$$k_1 = 0.316 \quad k_o = \text{nil}$$

$$k_2 = -0.204$$

[1]The selections were accomplished by means of a step-up procedure, as described in Appendix E.

These figures compare faborably with the $T \to \infty$ approximations ($k_1 = 0.321$, $k_2 = -0.198$) previously stated. The results show that further terms in the second order model were of little or no value in improving the more simple result. Although good results were obtained for both standard initial conditions, the adverse condition actually gave a lower error, approximately 25% of that for the ideal condition. This is to be expected because the basis function is a good approximation to the exact solution. In such cases the extrapolation errors from the adverse condition are no longer overriding, leaving the smaller initial error of condition two as the dominant factor in the improved result.

For the remaining discussion the more difficult short T problem is considered. Quite as expected, the second order control surface was, at best, effective only in limited regions of state space, as results from the ideal initial condition show. The results from the adverse condition (2) gave corresponding terminal errors several orders of magnitude greater, and therefore are not included in the plots. Despite its limitations as a good choice of basis function for the short T problem, the second order model results indicate that much can be done to improve the performance by giving careful attention to details in the fitting process. Thus, comparisons in Figure 8 show that for a fixed (narrow) size of data base, the larger number of data points gave a substantially better control function than the reduced number. Secondly, for a fixed number of data points some advantage was gained by reducing the data base size about the limited region of expected operation. The improvement occurred, however, at the expense of further performance degredation from initial conditions outside the narrow data base. A third observation regarding the second order model is that rather substantial improvements were made in the limited region of operation by a careful selection of the number and combination of terms employed. However these improvements did not extrapolate well outside of the small data base. Hence, it appears from this limited test that the labor involved in extensive searching for the best combinations may be justified mainly in special problems where operation is limited to "sufficiently small" regions.

It is interesting to observe that an increase in the number of terms, with ever higher least squares correlation, did not give uniformly better performance (compare the several flat sloped trends shown in Figure 8). This would seem to cast doubt on approaches based on indiscriminate use of a large number of terms, even when such complex functions could be afforded. Also note that very high correlation coefficients were obtained with just a few terms of the series (Table 2), but this did not assure acceptable performance as a control function (Figure 8, 9).

Results with the third order TVG control model show impressive improvement over the power series function, with reasonably good performance noted for both initial conditions (Figure 9). Thus it appears that efforts spent in establishing the correct form of control function should be given first priority, whenever possible, over attempts to wring out a more arbitrary selection. Simplified analytical solutions, previous experience with similar types of problems, and organized trial and error search, may be useful in these efforts (see Appendix E).

Figure 9 shows that poor results were obtained with the gain switched, linear TVG control model except in limited regions of operation (i.e., near initial condition one). Examination of the three sets of four segmented control coefficients showed that greater variations occurred in these terms as a function of the initial x_1, x_2 combination than occurred at the switching points along the time axis. Thus, the most significant result for this control function is that it is not well suited to this problem, even though it has the advantage of the TVG structure.

In overall summary, this case study affords a collection of empirical data which can be assessed in light of complete knowledge of the true feedback solution. One of the most important points observed is the illustration that goodness of fit in the sense of least squares may not assure goodness of fit in the sense of control system performance (terminal error plots of Figures 8, 9). Thus, little trouble is found in obtaining very high least squares correllations (e.g., $R^2 = 0.98$) in several different fittings of the same control function to the four dimensional optimal control hypersurface. Yet, as illustrated in the rms terminal error plots, the corresponding performance of two such cases from the same starting condition (x_o) could reasonably be judged to range from "quite good" to "totally unacceptable." The occurrence of this disparity in types of goodness of fit is expected, as discussed in Chapters I, III, and observed throughout the remaining case studies. Furthermore, the existence of this problem is justification for the need of a final simulation step in the design method (Step 5, Design Method Steps of Chapter I).

A second point well illustrated by this case problem is the undesireability of working with time varying gain controllers. While hardware mechanizations of this form of control may not always be difficult to implement, a possibly disturbing aspect is that the control becomes undefined after zero time-to-go, even though the desired terminal state may not yet have been reached. Thus, unless the problem requirements specifically demand it, ways to avoid this type of control will be useful. In the present case this result was effectively achieved by fixing T at some "large" number. In the next case study a similar effect is found by letting T be unspecified.

Case Two

(i) Analysis

The second case study considers the low order form of plant dynamics of Case One in cascade with a nonlinear gain function. The gain function includes a parameter (N) for changing the shape of the nonlinearity. By altering the value of N a set of problems are analyzed ranging from the linear gain case to a steeply rising saturation function as shown in Figure 10. Thus, the main purpose of this problem is to collect data on the design method as applied to various cases of nonlinear gain shapes, while retaining the second order dynamics for ease in presenting results through use of the phase plane, and for comparison with Case One.

The state and co-state differential equations for the open-loop trajectories are given in Appendix B. Due to the high degree of nonlinearity present with large N, the adjoint method was mainly ineffective as a means of solving the two point boundary value problem with realistic convergence rates. Hence the flooding technique was relied upon exclusively for generating the final data.

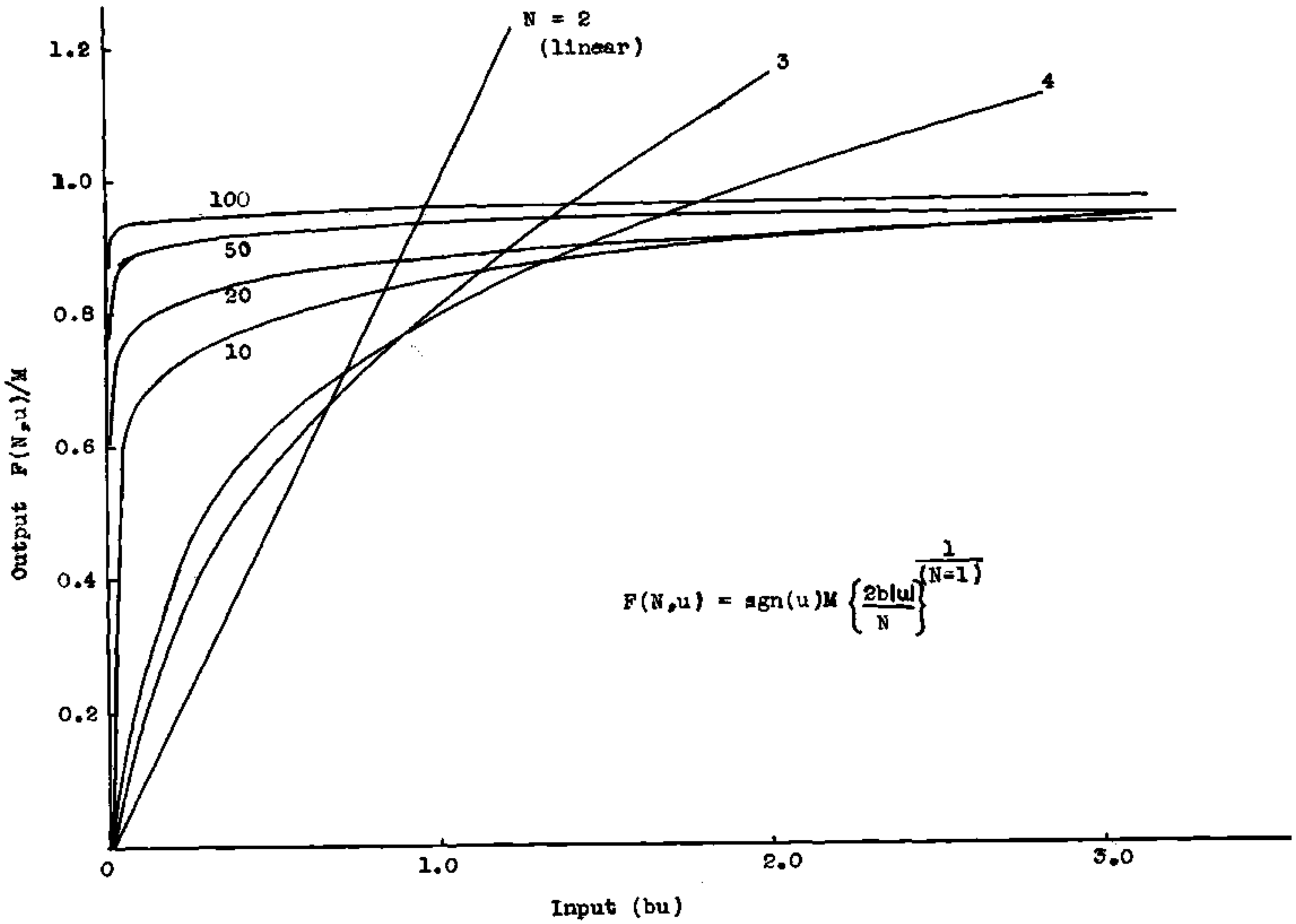

Figure 10. Case Two Nonlinear Gain Function.

Aside from the difference in gain function, the Case Two problem differs from Case One in the response interval T, which is now left unspecified. From the control equipment implementation standpoint this offers the advantage of a time independent control function, while the resulting response time is not required to be long. Under this condition the lowest possible cost J is incurred.

Again, using Bell's weight selection method the reasoning employed in the first problem could produce the same Q matrix for this case as a first selection. Therefore consider

$$Q = \begin{bmatrix} 0.1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since, however, this matrix is not positive semidefinite, sufficient conditions of optimality can no longer be assured for this nonlinear application (Chapter II). Indeed, a routine application to the case $N = 50$ yields the "looping" trajectories shown in Figure 11. Each intersection allows an alternate choice of path and response time T to the origin. Since T is free, not all these paths can be optimal unless the value of J is indifferent to the choices. Such is not the case here.

To avoid the oscillation problem a second weighting matrix is chosen such that Q is positive definite. In terms of a (ξ, ω_o) linear closed-loop characteristic root selection specification, Bell's equations for the present case would equivalently call for the diagonal elements

$$q_1 = \omega_o^4$$

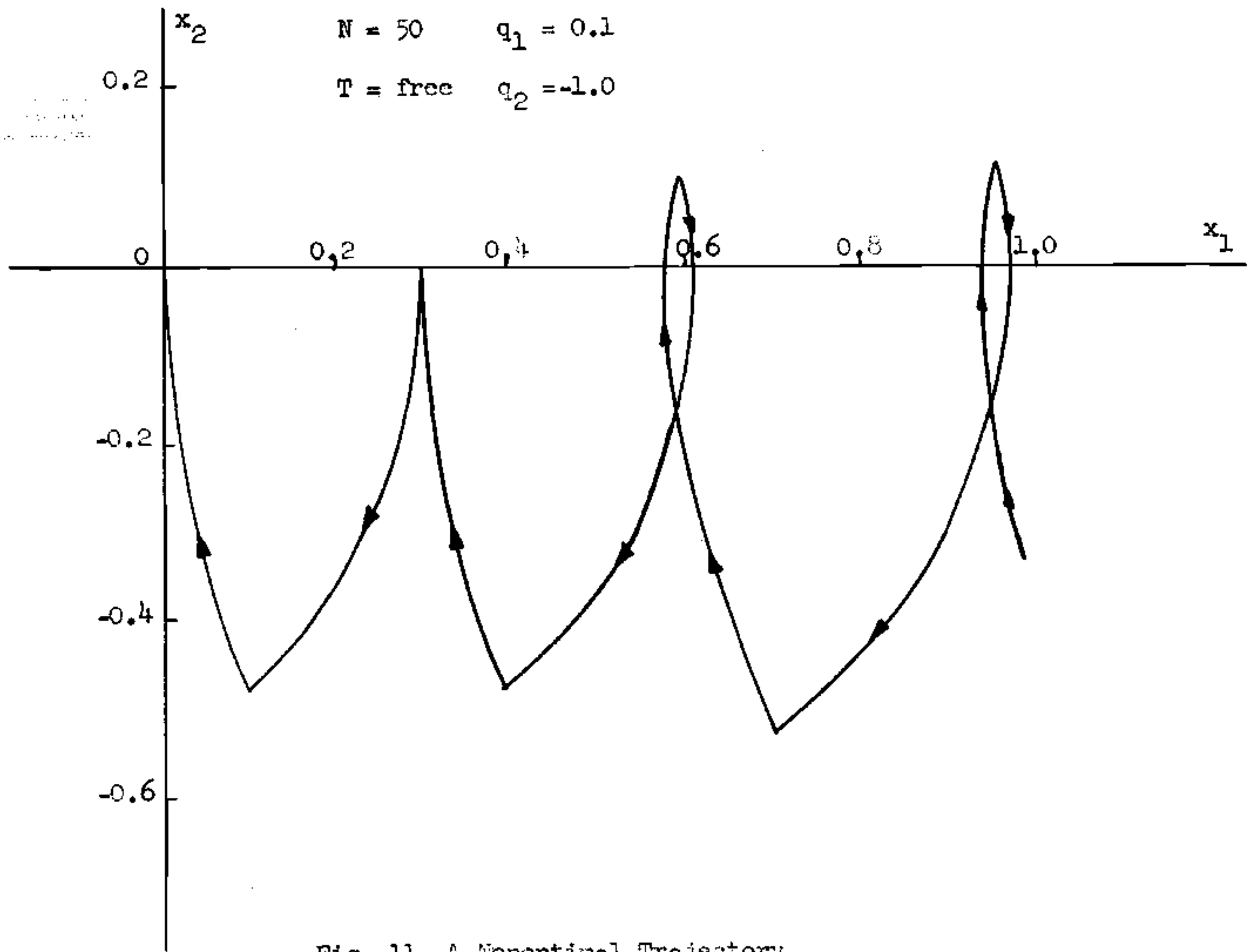

Fig. 11 A Nonoptimal Trajectory

$$q_2 = 2(2\xi^2 - 1)\omega_o^2 - \omega^2$$

Thus q_2 can never be positive for $\xi \leq 1/\sqrt{2}$ with the present form of plant. The implication of the sufficiency constraint on Bell's approach in the present case is that one must resort to a relatively sluggish choice of model root locations. Since in many cases it is unlikely that sluggish roots would be the designers original choice, the sufficiency constraint appears to have practical significance on the weight selection in this problem. This constraint does not always play a significant role. In the third case study computational considerations are overriding. The final selection for Q is taken as the positive definite diagonal matrix

$$Q = \begin{bmatrix} 625 & 0 \\ 0 & 13.91 \end{bmatrix}$$

The implied linear closed-loop characteristic root locations for these numbers correspond approximately to $\xi = 0.8$, $\omega_o = 5.0$ rad./sec.

An analysis is made on four distinct gain function models, corresponding to $N = 2, 3, 10, 50$ with T unspecified. Three basis functions are considered among these cases, each constrained to be an odd function about the origin (i.e., $\hat{u}(x) = -\hat{u}(-x)$) and to satisfy the nulling condition $\hat{u}(0) = 0$. The three control functions considered are

$$\hat{u} = -k^T x \ : \text{linear model} \tag{4.4}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$\hat{u} = -k^T x - x^T \psi x \quad : \text{ second order model.} \tag{4.5}$$

$$\psi = KS, \quad k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, \quad S = \begin{bmatrix} \mathrm{sgn}(x_1) & 0 \\ 0 & \mathrm{sgn}(x_2) \end{bmatrix}$$

$$\hat{u} = -(x^T K x)(x^T \delta) \quad : \text{ third order model.} \tag{4.6}$$

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, \quad \delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The linear model (4.4) was chosen on the basis of simplicity, allowing the most desirable form of hardware if acceptable performance can be achieved. This model is applied to all four cases of gain function. The second order model (4.5) is applied to the case $N = 3$, and allows the next higher order terms to be included. The signum functions (sgn) are necessary to maintain the odd function constraint. The third order model (4.6) is applied to the cases $N = 10, 50$. In these cases the shape of the nonlinearity tends toward a rough approximation of an ideal relay. The choice of (4.6) over (4.5) as a higher order model for the latter two nonlinear cases is based on some analytical results of Bass and Webber (16) for a somewhat related problem. In their paper they considered bang-bang control of a linear plant based on the quadratic performance index, and showed that the first higher order terms in a series expansion of the control law were cubic rather than quadratic in form.

In addition to the least squares derived control functions (4.4, 4.5, 4.6), a linear control function is also derived analytically and tested on each of the four cases for comparative purposes. The basis for the analytical solution is simply to compute the linear gains required to achieve a selected set of closed-loop root locations with the linear gain model (i.e., N = 2), without carrying through any cost index optimization. The selected root locations were taken as the same linear model root locations implied by the weight selection Q, i.e., roots which yield the linear closed-loop parameters $\xi = 0.8$, $\omega_o = 5.0$ rad./sec.[1] This calculation results in the linear "analytical control"

$$\hat{u} = -c^T x \;:\; \text{analytical control} \tag{4.7}$$

$$c = \begin{bmatrix} 25.0 \\ 7.7 \end{bmatrix}$$

As summarized in Table I, all the test results for this problem are displayed in terms of phase plane plots, and comparative plots of J_ε, T_ε, t_r, t_s (as defined at the beginning of this chapter) versus initial conditions. The terminal error criterion ε was taken as a small square of magnitude 0.002 about the origin in state space. Since all state and control responses are symmetrical about the origin, only half of the response surfaces are actually used and plotted. Using the stated performance measures and the control functions (4.4, 4.5, 4.6, 4.7), the

[1] Therefore the control is also identical to the optimal control with the selected Q, for the case $T \to \infty$.

specific tests carried out on each of the four gain cases are now presented.

N = 2:

Results for the two linear control functions (4.4, 4.7) are compared. Figure 12 gives the phase plane plots of the optimal data base and the two control simulations. Figure 13 compares the performance parameters as determined from various initial conditions. Due to its similarity with Case One and the relative success of the results, no data base or outside initial condition variations are considered.

N = 3:

Results are analyzed for the linear and second order control functions (4.4, 4.5, 4.7). Figure 14 shows phase plane comparisons based on a relatively narrow data base from the half surface. An outside initial condition test is included. Figure 15 gives second order control comparisons based on a wider size data base. Figure 16 gives the corresponding performance parameter comparisons from various initial conditions.

N = 10:

A phase plane comparison is given in Figure 17 for the linear and cubic control functions (4.4, 4.6, 4.7) as computed from the single data base shown. Figure 18 gives the corresponding performance parameter comparisons.

N = 50:

Phase plane comparisons are given in Figure 19 for the linear and cubic control functions (4.4, 4.6, 4.7) for the particular data base shown. Figure 20 gives further phase plane comparisons with the higher

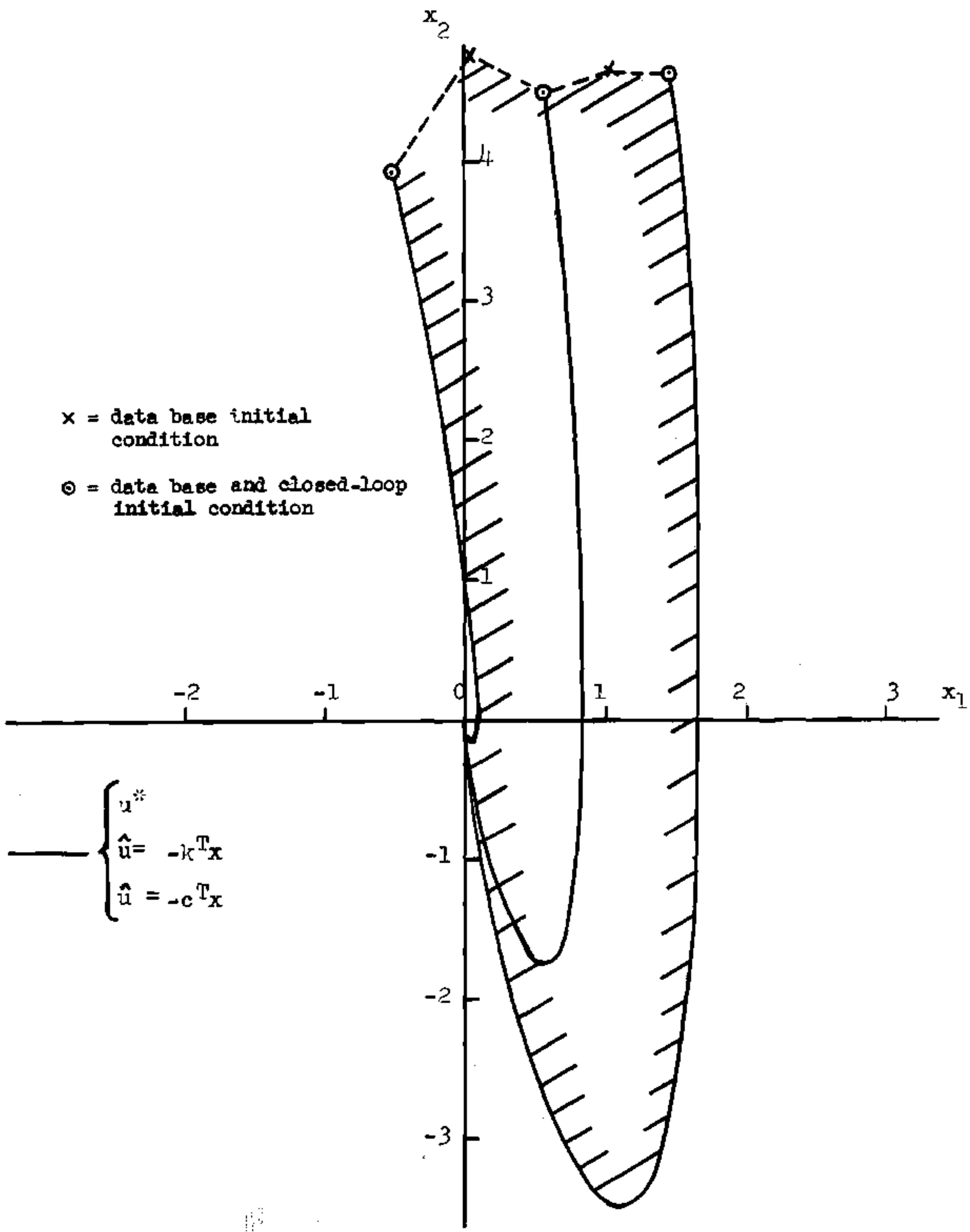

Figure 12. Case Two Phase Plane (N = 2).

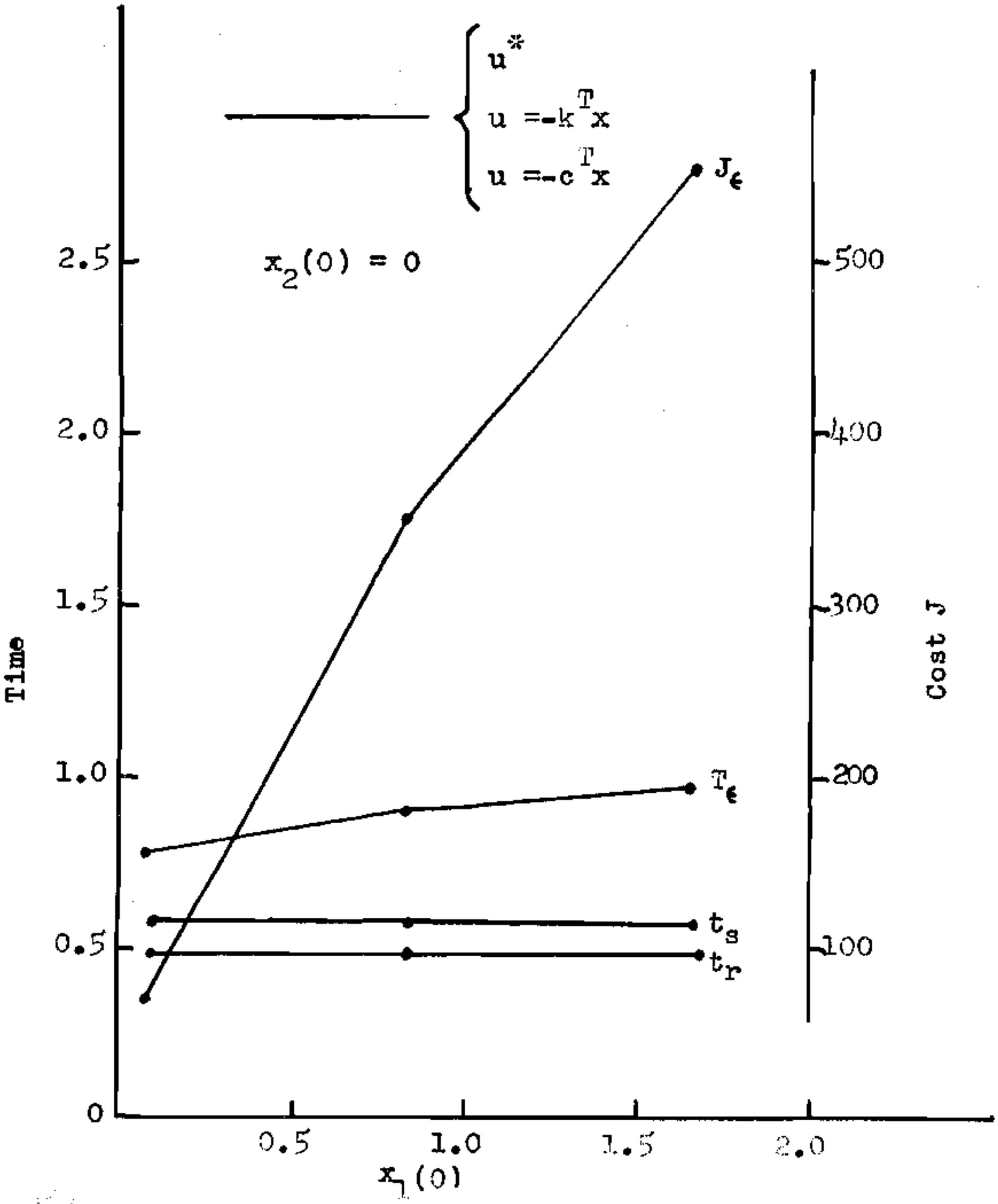

Fig. 13 Performance Parameters (N=2)

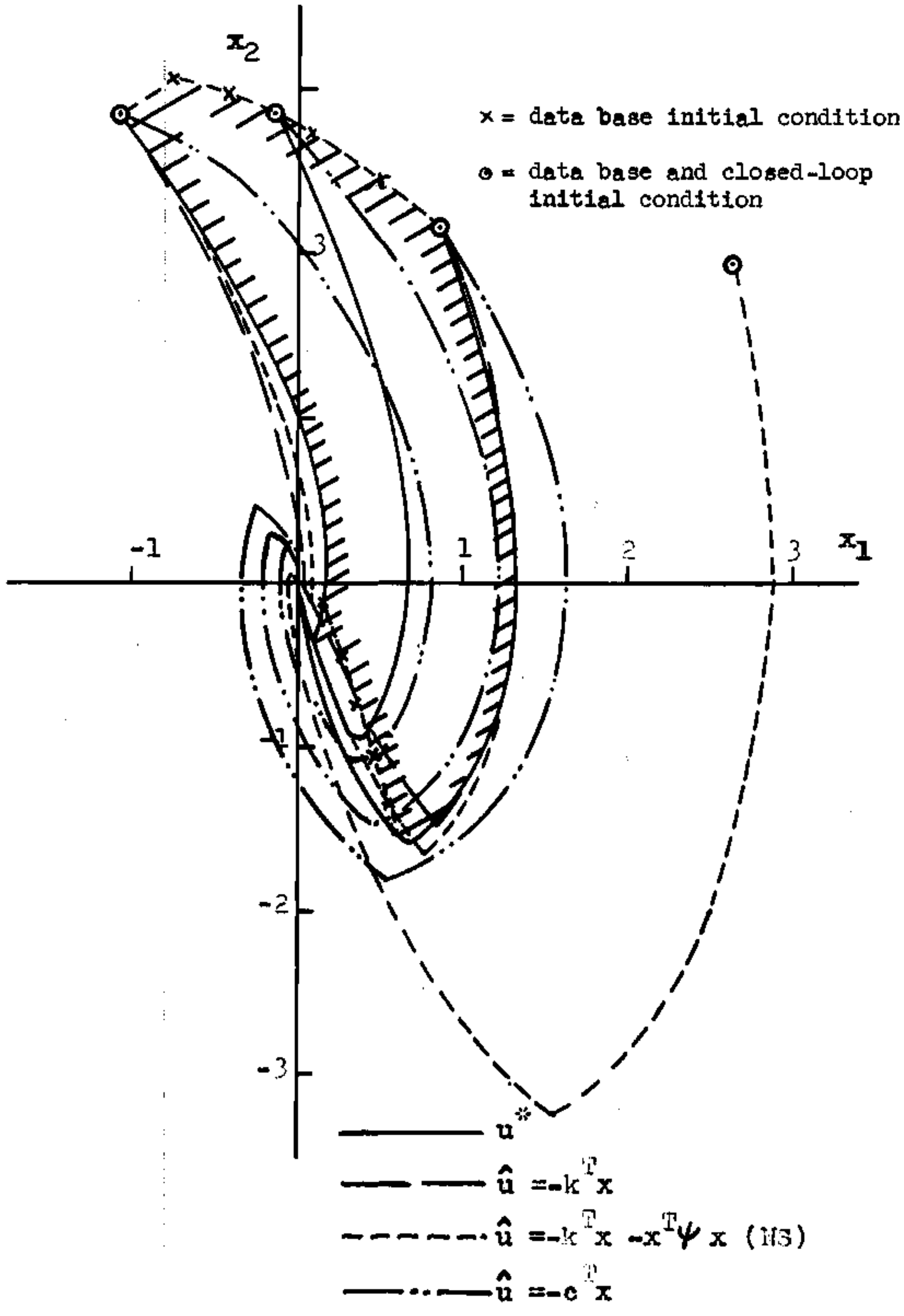

Fig. 14 Case Two Phase Plane (N = 3,NS)

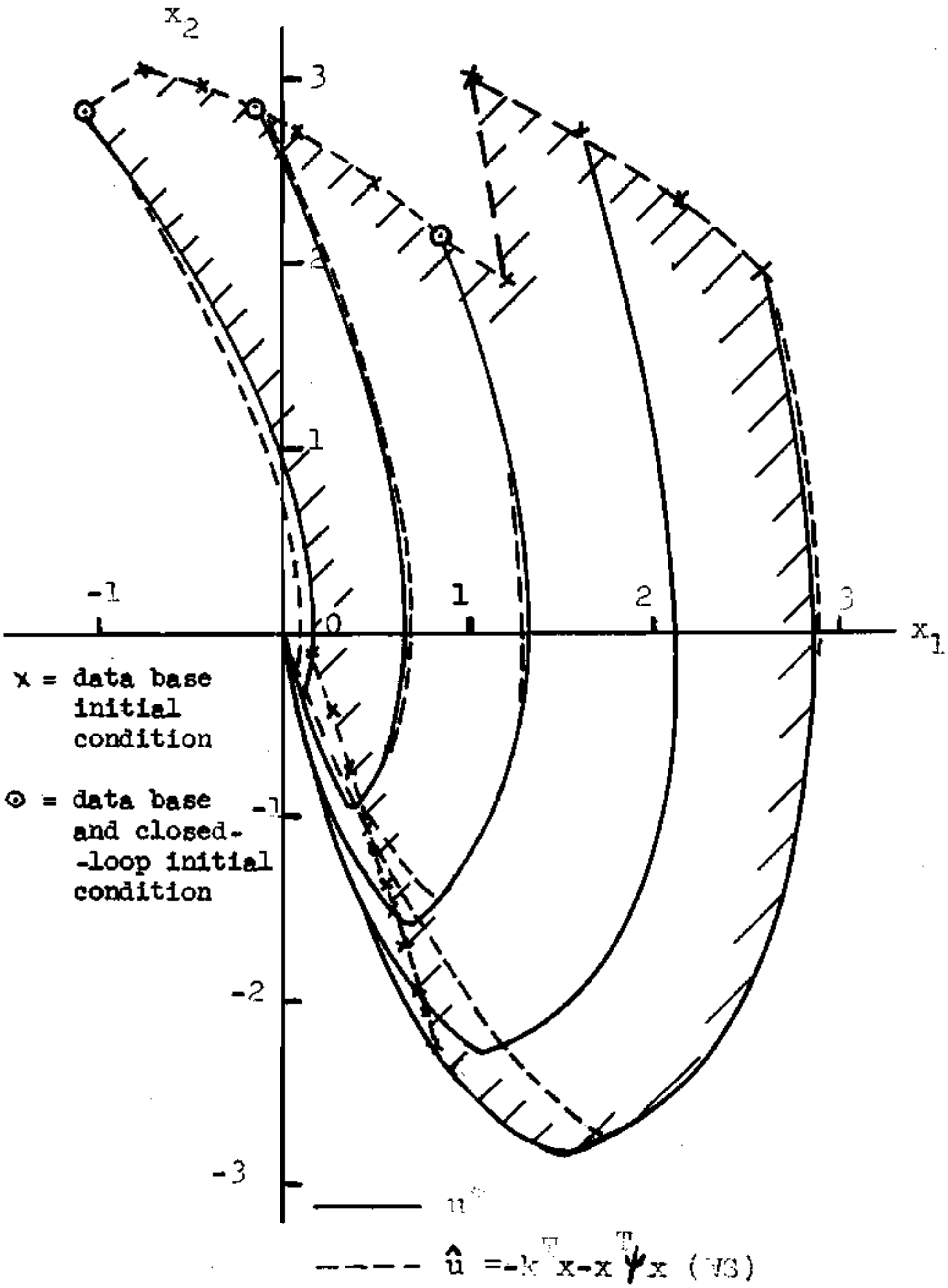

Fig. 15 Case Two Phase Plane (N = 3,WS)

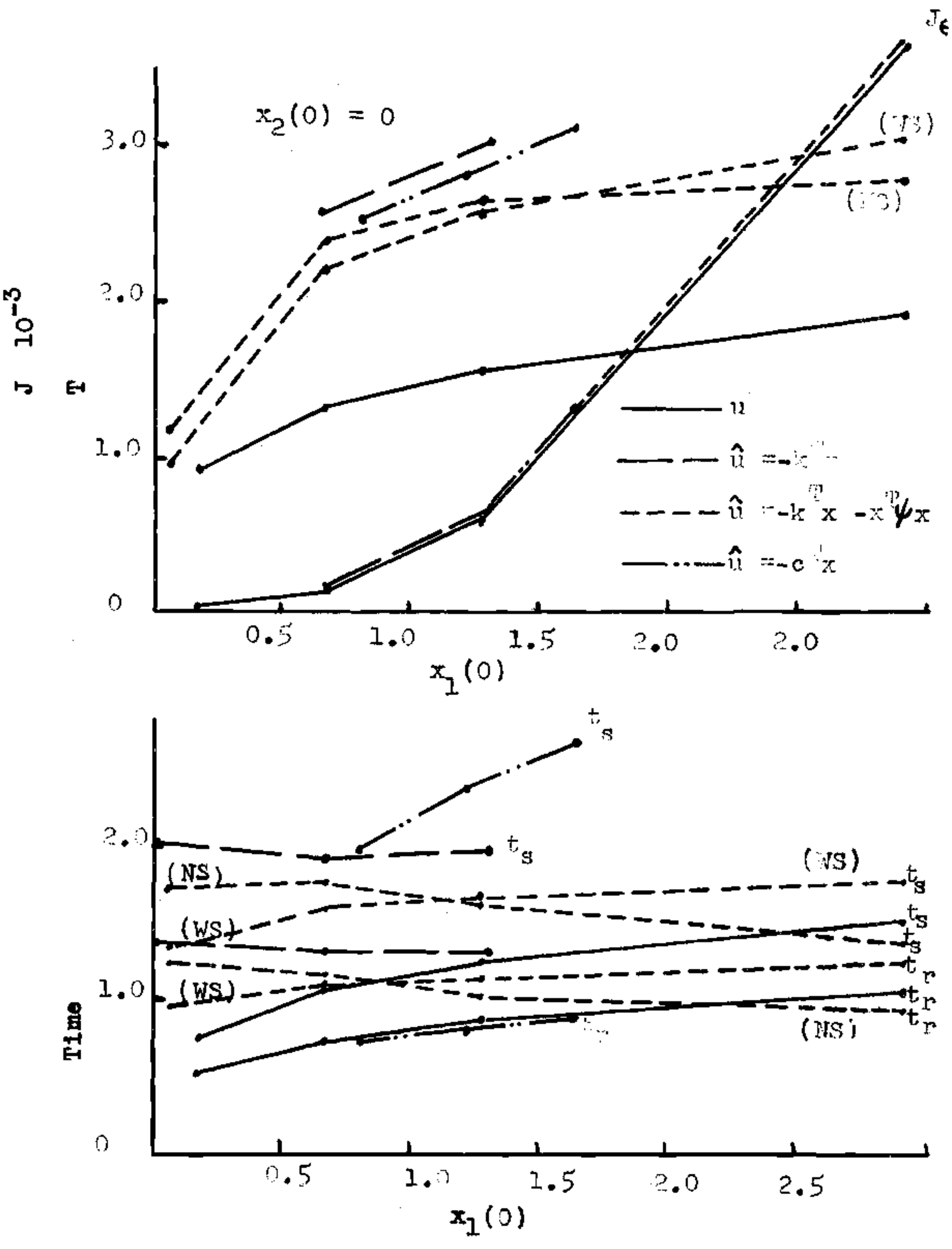

Fig. 16 Performance Parameters (N = 3)

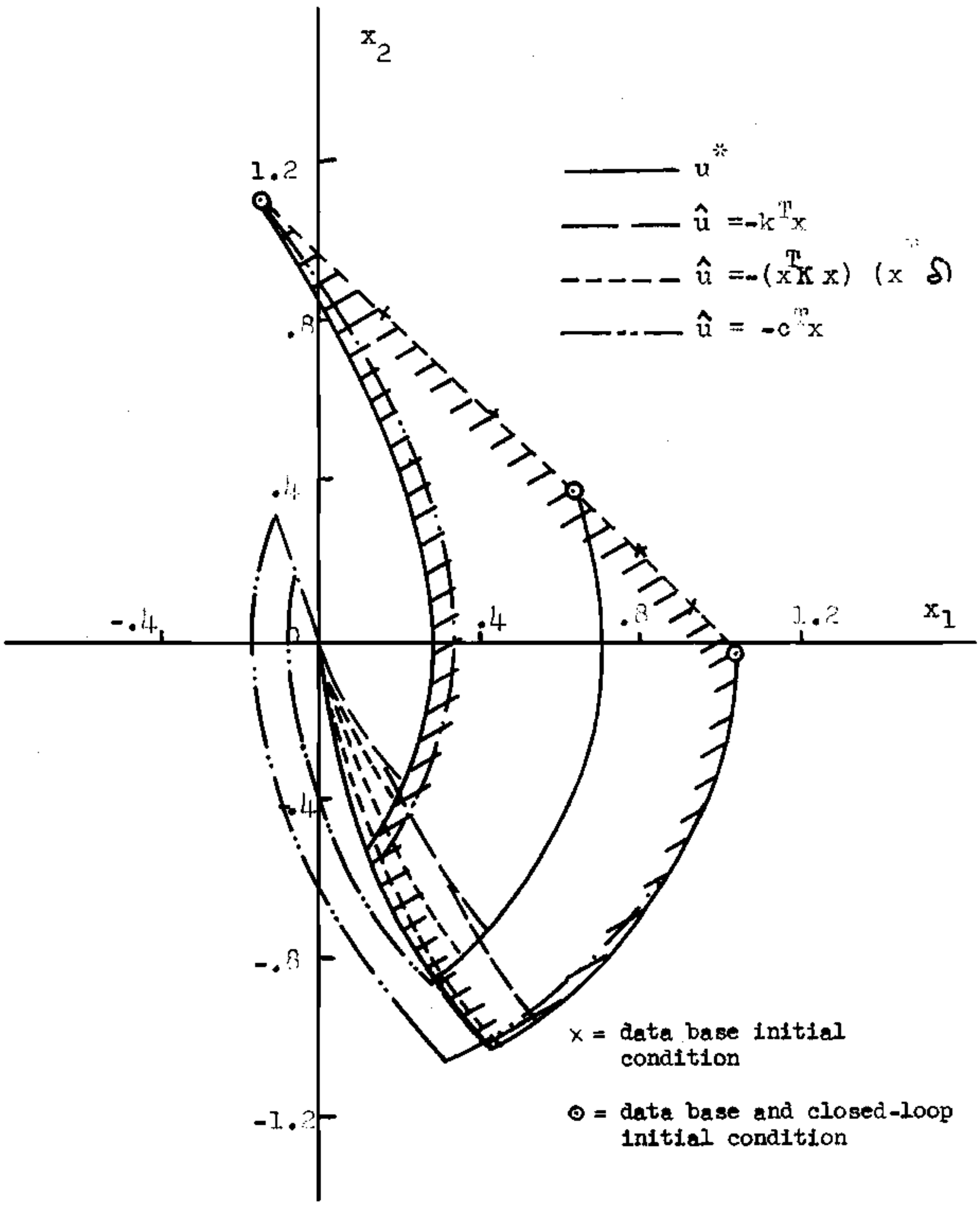

Fig. 17 Case Two Phase Plane (N = 10)

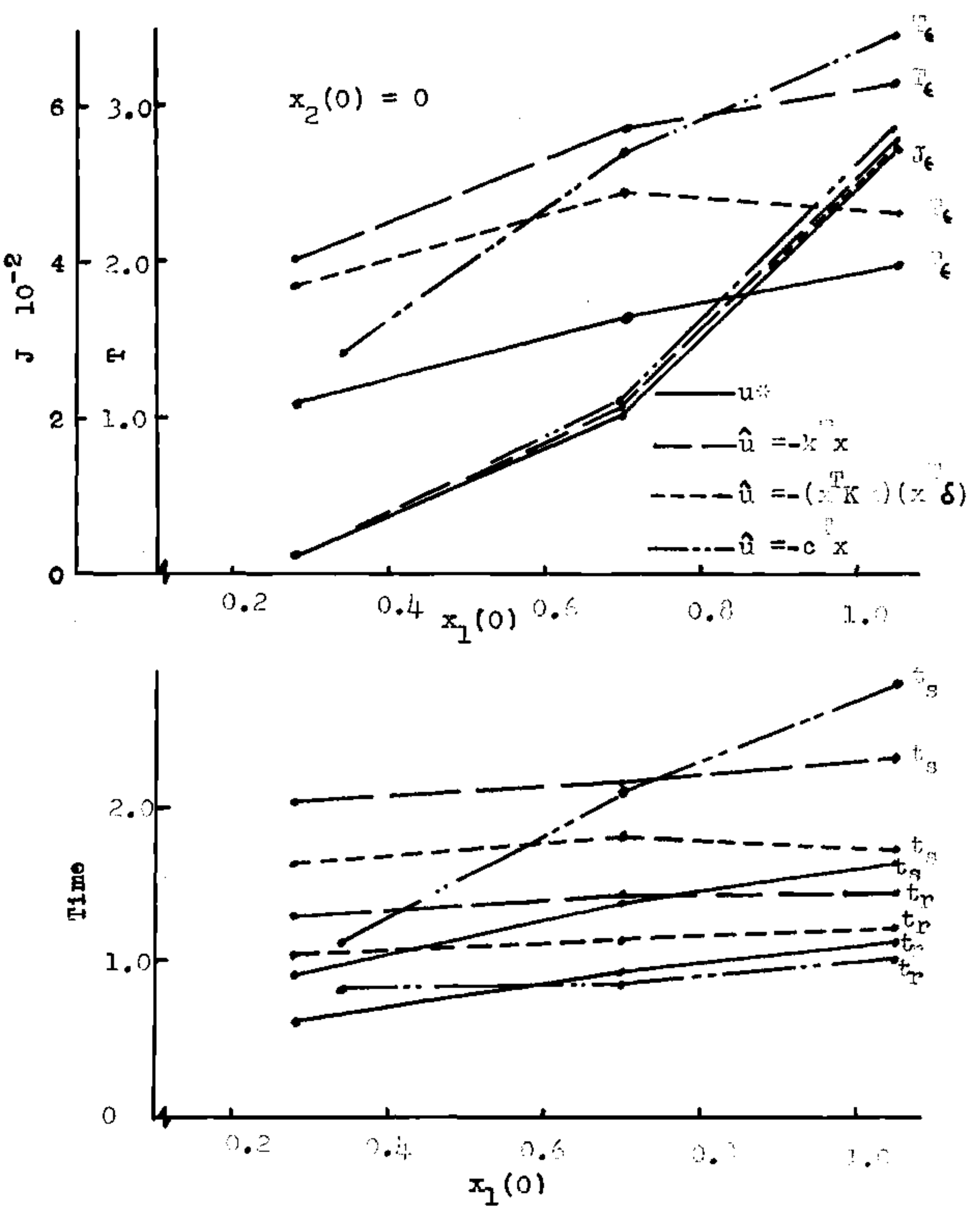

Fig. 18 Performance Parameters (N = 10)

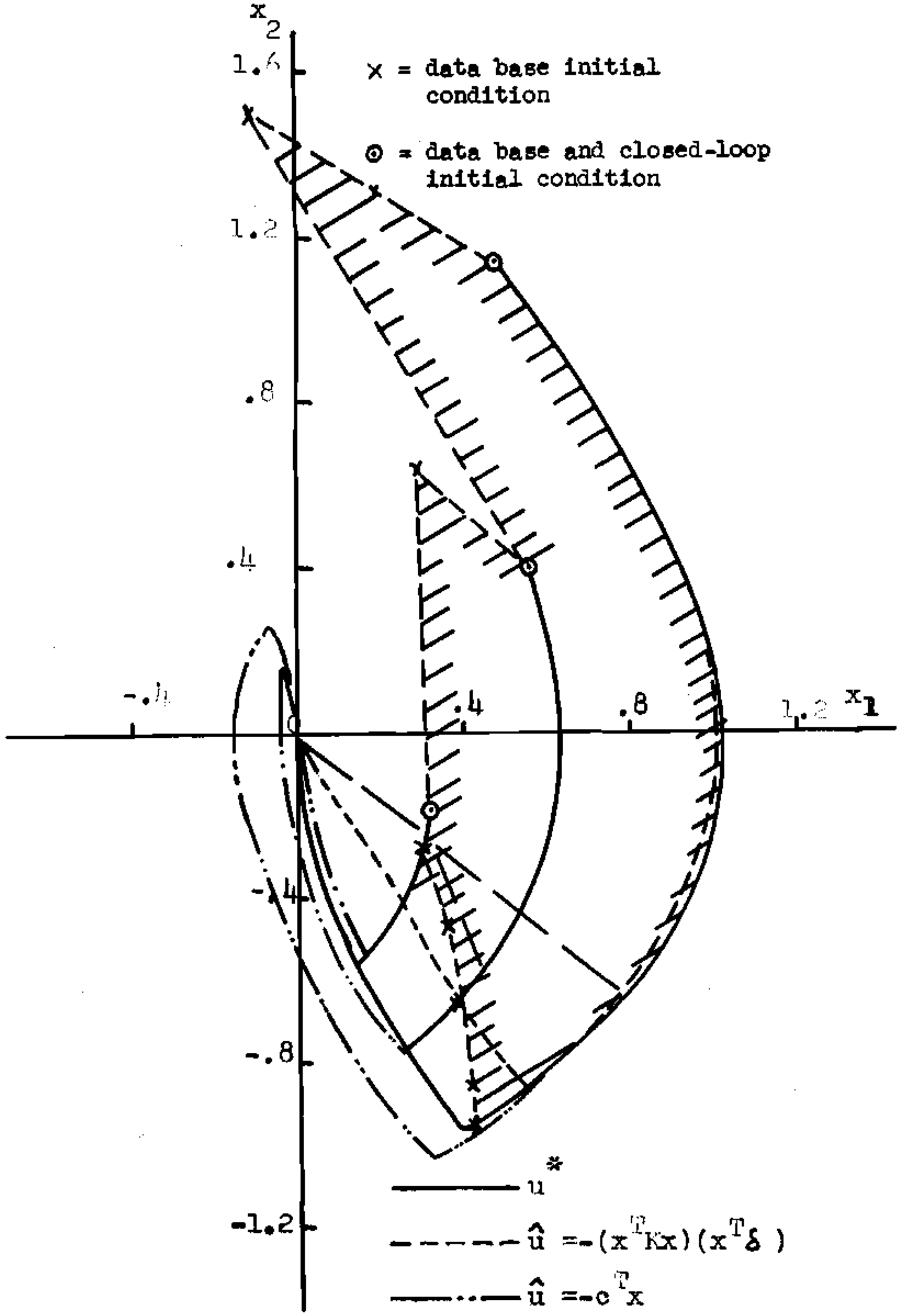

Fig. 19 Case Two Phase Plane (N = 50)

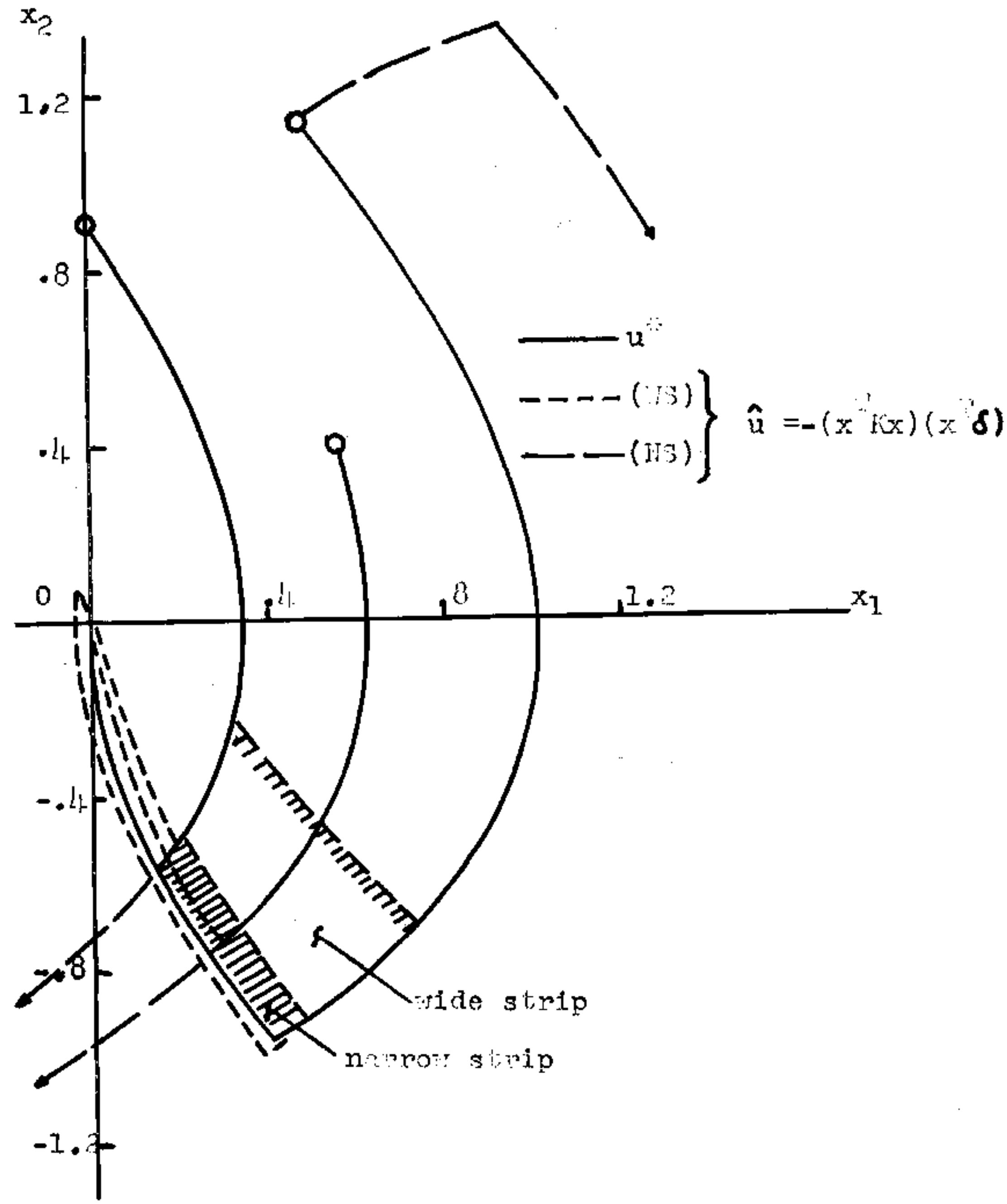

Fig. 20 Case Two Phase Plane (N=50, data weighting)

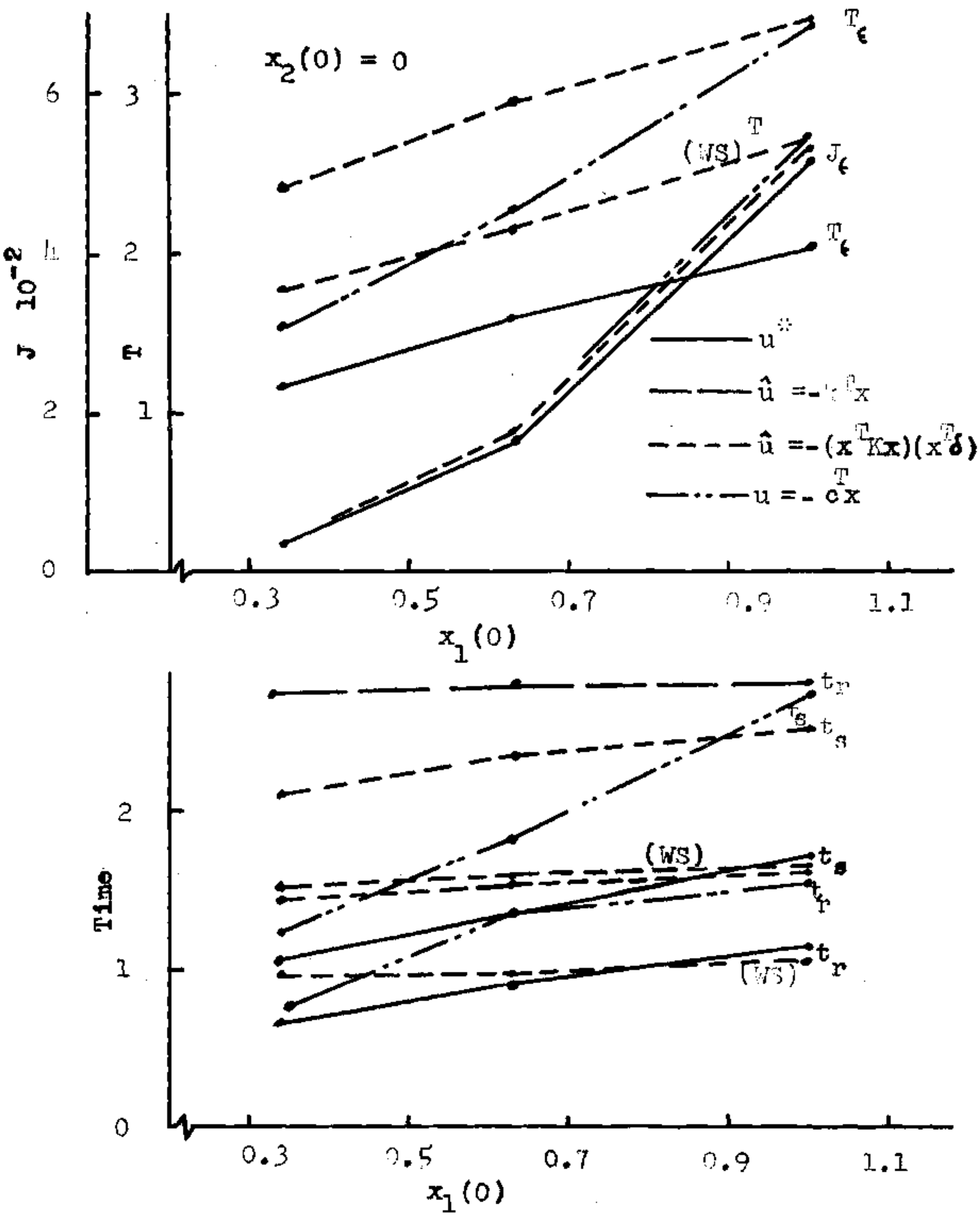

Fig. 21 Performance Parameters (N = 50)

order control derived from several data base weighting attempts as indicated. The corresponding performance parameter plots are given in Figure 21.

(ii) Discussion

For the linear gain case ($N = 2$), the phase plane plots (Figure 12) show essentially identical results between the optimal response, the least squares linear control approximation (4.4), and the analytical control (4.7). As noted, the latter is identical to the optimal solution for the selected Q and a fixed response time $T \to \infty$. It therefore appears that in this linear problem the unspecified T case behaves essentially (though not exactly) the same as fixing T at a large number. This is further illustrated by comparing the linear gains derived analytically for the control (4.7) with the least squares derived gains for the linear control (4.4), based on the unspecified T numerical data. The gains for the latter case turned out to be

$$k_1 = 24.985 \qquad (R^2 = 0.999588)$$

$$k_2 = 7.749$$

which compare closely with the two constants in (4.7). Figure 13 illustrates the invariant nature of t_r and t_s in the linear problem.

Consider next the gently curving nonlinear gain case $N = 3$. The phase plane results for the narrow size data base show (Figure 14) the adequate degree of success achieved with the least squares linear (4.4) and second order control functions (4.5). The latter case fills in some of the fine detail absent from the former, but both stable responses would probably often be considered acceptable. The analytical control is seen to give more oscillatory responses than the least squares and optimal controls. This can be attributed to the higher effective small signal gain of the nonlinear element over that of the linear ($N = 2$) case. Since no attempt was made to compensate the intentionally simple, analytically derived control to the small signal gain changes (whose effective values depend in part on the initial conditions) it is not surprising that the relative stability of the analytical control decreases somewhat for larger N.

The second order least squares control becomes more oscillatory from the initial condition outside the data base, but the degradation does not appear to increase rapidly with the extrapolation. The wide size data base, with roughly the same large number of data points, gives the response shown in Figure 15. As could be expected, the response from the initial condition outside the narrow data base is improved by the fitting from the wider base. This was achieved with only slight loss in accuracy in the narrow base region.

Figure 16 shows the optimal responses are largely superior in terms of t_r, t_s, T_ε, over the other control functions. The analytical control is at its best in terms of rise time t_r. This could be expected from the more oscillatory nature of its response. Somewhat surprisingly, the narrow data base size gives improved T_ε, t_r, and t_s from the outside initial condition, as compared to the results for the larger size fitting. Thus, it may not always be certain that the added costs of a larger data base will yield a desired form of improvement. The costs J_ε are seen to be somewhat near that of the optimal responses for all control functions.

The more sharply curving gain case $N = 10$ can be seen from Figure 17 to result in a moderately sluggish least squares linear response while the linear analytical control is again on the oscillatory side. The third order control function is especially good at the larger distances from the origin, while closer in it tends to approach the performance of the linear function. The latter observation on the cubic control function did not appear to carry over to the more sharply nonlinear case of $N = 50$.

Figure 18 shows that the optimal responses for $N = 10$ are generally superior in terms of T_ε , t_r , t_s over the other control functions. The analytical control has a fast rise time and also gives relatively low T_ε , t_s for the small initial conditions. As would hopefully be the case, the higher order model is superior to the linear function in terms of these response time measures. All control functions give similar cost (J_ε) figures, though in itself this confirmation would not be considered of overriding importance.

The phase plane results for the very sharply breaking nonlinear gain shape $N = 50$ imply an unusually sluggish response with the least squares linear control for the data base given in Figure 19. The third order model gives some improvement but is still rather sluggish. The analytical control is fairly close to the optimal result for two out of the three trajectories shown.

The sharply breaking shape of the $N = 50$ gain function poses a rather special situation, since the output of the function is fairly constant at all input magnitudes except very near zero, where rapid changes occur. Thus, it could be expected that errors in the least squares fitting at small control levels could be critical, while large fitting errors could be tolerated elsewhere. A second feature to note regarding the response of this sharply breaking gain function is that they behave much like an ordinary switching system. Hence all optimal trajectories tend to merge at a "switching line" where the control reverses, causing a common path to be followed to the origin. Thus, it would seem that improved control could be obtained by concentrating more of the data base in the region where the zero crossing occurs, but yet not on the common path to the origin since on this segment the control can no longer be unique.

The phase plane results for two cases of this suggested data weighting on the third order control function are shown in Figure 20 to give substantially differing responses. Thus, the fitting from the very narrow data strip near the control zero crossing gives an unusually poor diverging response near the origin. However, by taking the data strip somewhat wider as shown, the response turns out to be the best of all those tried.

Figure 21 shows that the performance parameter comparisons follow the same general pattern as the results for the $N = 10$ gain function. The advantage of the data weighting on the third order control is clearly evident in these performance measures.

A number of comparisons between the results for the four gain function cases can be made. In the nonlinear gain problems the optimal responses are largely superior in terms of the T_ε, t_r, t_s, performance measures when compared to the other control functions. This lends some confidence to the use of the quadratic performance index as an acceptable guide to design. It is observed that as N is increased, the T_ε, t_r, t_s, performance measures take on longer values. This is due to the increasing effect of the saturation, which limits the available output magnitudes for each level of input. The cost J, conversely, tends to become smaller as N is increased, for low values of the initial condition (x_o). This is because of the high efficiency available from the large N type of nonlinearity, i.e., a small u (and hence u^2) is all that is needed to produce essentially a full output magnitude. For large values of x_o the opposite effect again holds, i.e., J_ε increases with increasing N. The reason for the reversal in trend is due to the limited output levels of the nonlinear element for large N. This causes the response times from the large x_o starting conditions to be significantly longer, with corresponding increase in the integrated cost J_ε.

Quite as expected, the results show that the higher order control functions give the best performance. Whether the amount of improvement over the linear functions are worth the added control equipment cost would have to be judged in relation to the needs of the particular application. However, the results presented here should be indicative of what to expect in related situations. Similarly, the inclusion of the analytical control function results offers, to a small degree, some opportunity for comparison of the design method with those of an alternate approach. In this connection it should be noted that the relatively crude analytical control results could surely be improved upon merely by the use of a few trial and error adjustments. As such, the very simple means used to derive them could be considered to have yielded something of a lower bound to the performance from an alternate approach.

It is noted that the burdening costs of optimal control theory and multivariate least squares analysis do not appear competitive with the alternate approach (given some trial and error adjustment) on a problem of this relative simplicity. The primary contribution of this case study is therefore found in the accumulation of empirical experience regarding the application of the method.

This case study points out several practical aspects regarding the use of the method. Thus, an example of the sufficiency constraint on the weight selection method of Bell has been noted. The exceptional computational difficulty in the direct solution of the two point boundary value problem with highly nonlinear functions is observed. The value of the flooding technique as a means of data base generation is emphasized. The need for careful attention to aspects in the least squares fitting (i.e., proper basis function and data base selection) are again observed in relation to differences in the nature of the plant.

Case Three

(i) Description

A fourth order, type two tracking servo is considered for the third case study. The two integrations in the forward loop of the servo make it especially difficult to stabilize. Thus, the problem presents a rather severe test of the method as a design procedure.

A mechanization drawing of the system is shown in Figure 22.

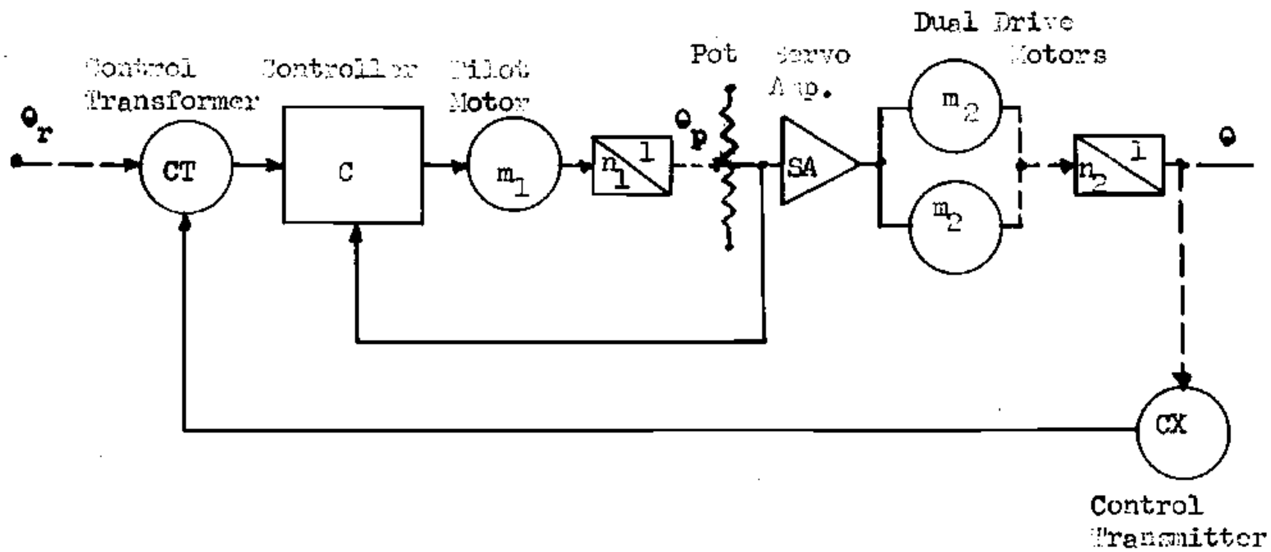

Fig. 22 Case Three Mechanization Drawing

The pilot motor positions the pot θ_p such that the dual drive motors can match a steady command speed $\dot{\theta}_r$ with no steady state error. The drive motors operate in tandem, and are biased slightly against each other to remove backlash from the output gearing. Three two phase servomotors are used. A linear dynamic model represents the lightly loaded pilot motor. The output drive motors include the common torque-speed-volt nonlinearities characteristic of the two phase motor. A linearized option for the drive motors is also included in the analysis.

In terms of a cascade representation of unilateral block elements the system can be viewed as a set of linear pilot motor dynamics feeding the drive motor nonlinear process. The pot acts as a buffer, allowing only unilateral flow of the signal. The pilot and load motor outputs and their time derivatives are selected as a basis for the four state variables. This selection allows all state variables to be measured with readily available position and velocity transducers. The state variables are further devised such that the steady state operating condition is a null signal in all four components. This results in the state vector definition

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta - \theta_r \\ \dot{\theta} - \dot{\theta}_r \\ \theta_p - \theta_{p_r} \\ \dot{\theta}_p \end{bmatrix} \qquad \begin{aligned} &\dot{\theta}_r = \text{constant speed reference input.} \\ &\theta_{p_r} = \theta_p \Big|_{\dot{\theta}=\dot{\theta}_r} = \text{constant.} \end{aligned}$$

A complete mathematical description of the problem is given in Appendix C, including the nonlinear motor representation and the selection of a typical set of plant design constants. All three motors were assumed selected as a common stock item. However, as will be shown below, the exceptionally high co-state sensitivities of this problem make solution of the open-loop trajectories impossible without a resort to double precision[1] arithmetic or a modifying assumption, in all but very short response interval (T) cases. To relieve this difficulty the modifying assumption is made that the pilot motor damping is negligible in most of the considered examples, as indicated.

<u>(ii) Selection of Q and T</u>

Bell's method, as discussed in Chapter II, is again applied to the weight selection (Q) problem. As suggested in Reference 49[2], Whiteley's standard forms are used to select the closed-loop root locations for the model fourth order linear response. The applicable form is

$$s^4 + 7.2\omega_o s^3 + 16\omega_o^2 s^2 + 12\omega_o^3 s + \omega_o^4 = 0$$

where the "bandwidth" ω_o is here chosen as 5 rad./sec. In factored form this gives the characteristic equation

$$(s + 0.48)(s + 7.82)(s + 8.80)(s + 18.90) = 0$$

Thus, all roots turn out to be on the negative real axis.

Now, per Table 2 of Bell the appropriate <u>diagonal</u> matrix elements c_{ii} are computed from the relations

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[1]Single precision on the Burroughs B-5500 carries twelve significant digits.

[2]pp. 571.

$$c_{11} = \omega_a^2\omega_b^2\omega_c^2\omega_d^2 - b_1^2$$

$$c_{22} = \omega_a^2\omega_b^2\omega_c^2 + \omega_a^2\omega_b^2\omega_d^2 + \omega_a^2\omega_c^2\omega_d^2 + \omega_b^2\omega_c^2\omega_d^2 + 2b_1b_3 - b_2^2$$

$$c_{33} = \omega_a^2\omega_b^2 + \omega_a^2\omega_c^2 + \omega_a^2\omega_d^2 + \omega_b^2\omega_c^2 + \omega_b^2\omega_d^2 + \omega_c^2\omega_d^2 + 2(b_2b_4 - b_1) - b_3^3$$

$$c_{44} = \omega_a^2 + \omega_b^2 + \omega_c^2 + \omega_d^2 + 2b_3 - b_4^2$$

where the ω's are the closed-loop root selections and the b_i are appropriate linearized plant constants. Taking the case where the pilot motor damping (D_1) is not negligible gives the result

$$c_{11} = 3.890 \times 10^5 \text{ sec.}^{-8}$$

$$c_{22} = 1.703 \times 10^6 \text{ sec.}^{-6}$$

$$c_{33} = 4.883 \times 10^4 \text{ sec.}^{-4}$$

$$c_{44} = -19.30 \text{ sec.}^{-2}$$

This choice is not positive semidefinite and hence sufficient conditions of optimality are not assured. By invoking the assumption $D_1 = 0$, or by other means, c_{44} can be made positive and the sufficiency constraint satisfied.[1] It is found, however, that the co-state sensitivities exert an overriding influence in this problem. Thus,

[1]It is remarked, however, that several other logical root selections, based on the methods discussed in Reference 50, gave similar nonpositive Q, illustrating that the result can often be desirable. Thus, note (2.10) and the unnecessary role of optimality in the problem.

selection of a modified alternate, meeting adequate response and computational constraints, is deferred temporarily, and development of the present selection is continued in its original form for later comparisons. Since the selected state variables are not phase variables, Q is computed from (2.14)

$$Q_c = \frac{1}{K^2} M^T C M$$

where M is a linear transformation of the form y = Mx, as discussed in Chapter II. In Appendix C it is shown, however, that the linearized transformation of this problem is of the modified form y = Mx + b (C-19). Moreover, an attempt to derive the equivalent of (2.14) with the added vector b yields a cost index with terms linear in the state, thus destroying the quadratic property of the index. This difficulty with the Q selection approach in this problem can be avoided by setting b = 0. This corresponds to the case $\theta_r = \dot{\theta}_r = 0$ (C-20).

On the above basis the matrix multiplication (2.14) gives for the Q_c matrix

$$Q_c = \begin{bmatrix} 2.90 & 0 & 0 & 0 \\ 0 & 16.51 & -1.559 & -.0213 \\ 0 & -1.559 & 0.629 & .0866 \\ 0 & -.0213 & .0866 & -.00026 \end{bmatrix} \qquad (4.8)$$

To assess the validity of the weight selection (4.8), results are compared in Figures 23, 24 with alternate selections Q = diag. (Q_c) (omission of off diagonal terms) and Q = 0 (the simplest possible

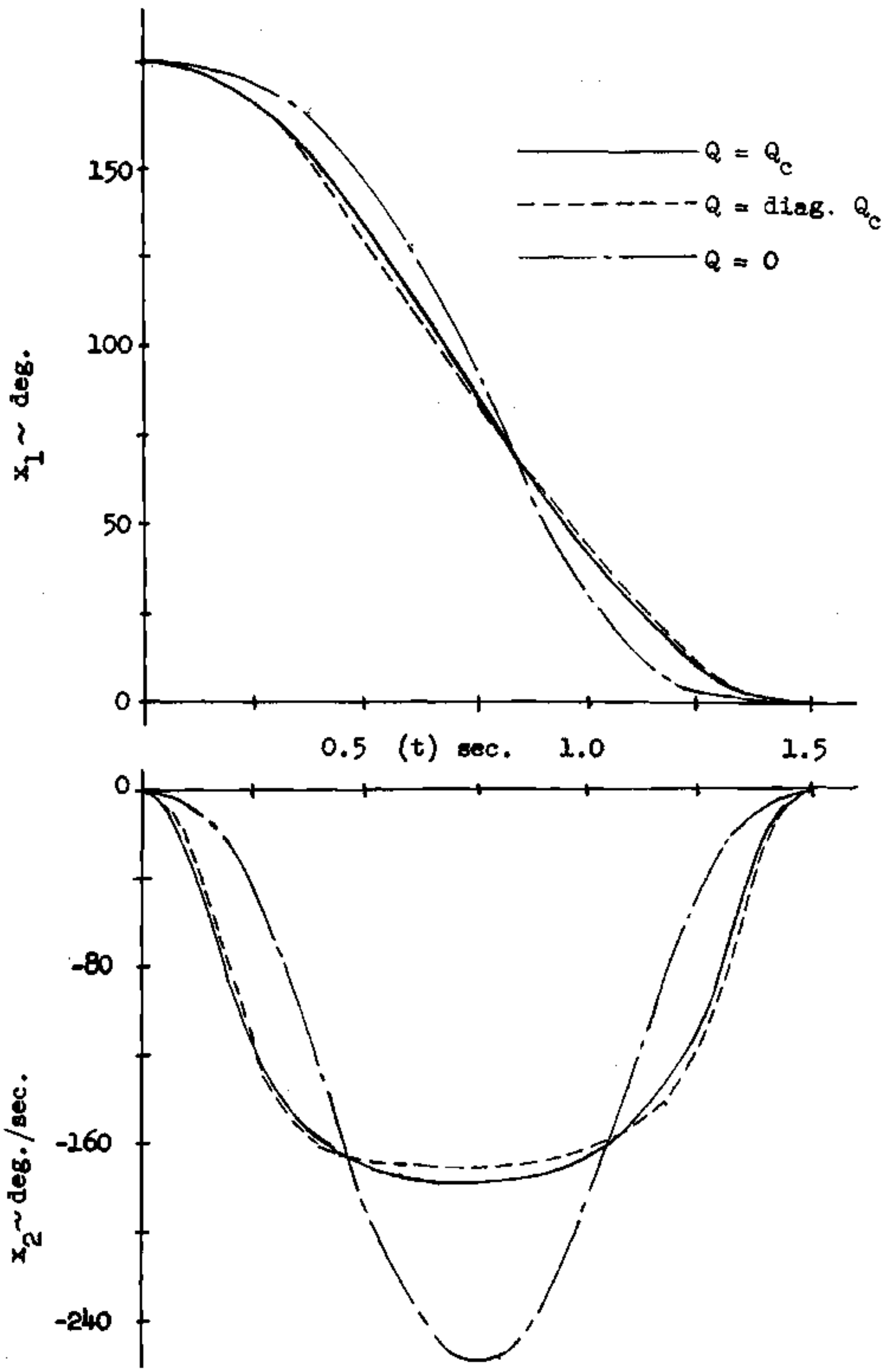

Fig. 23 Open-Loop Q Comparisons (x_1, x_2) For T = 1.5

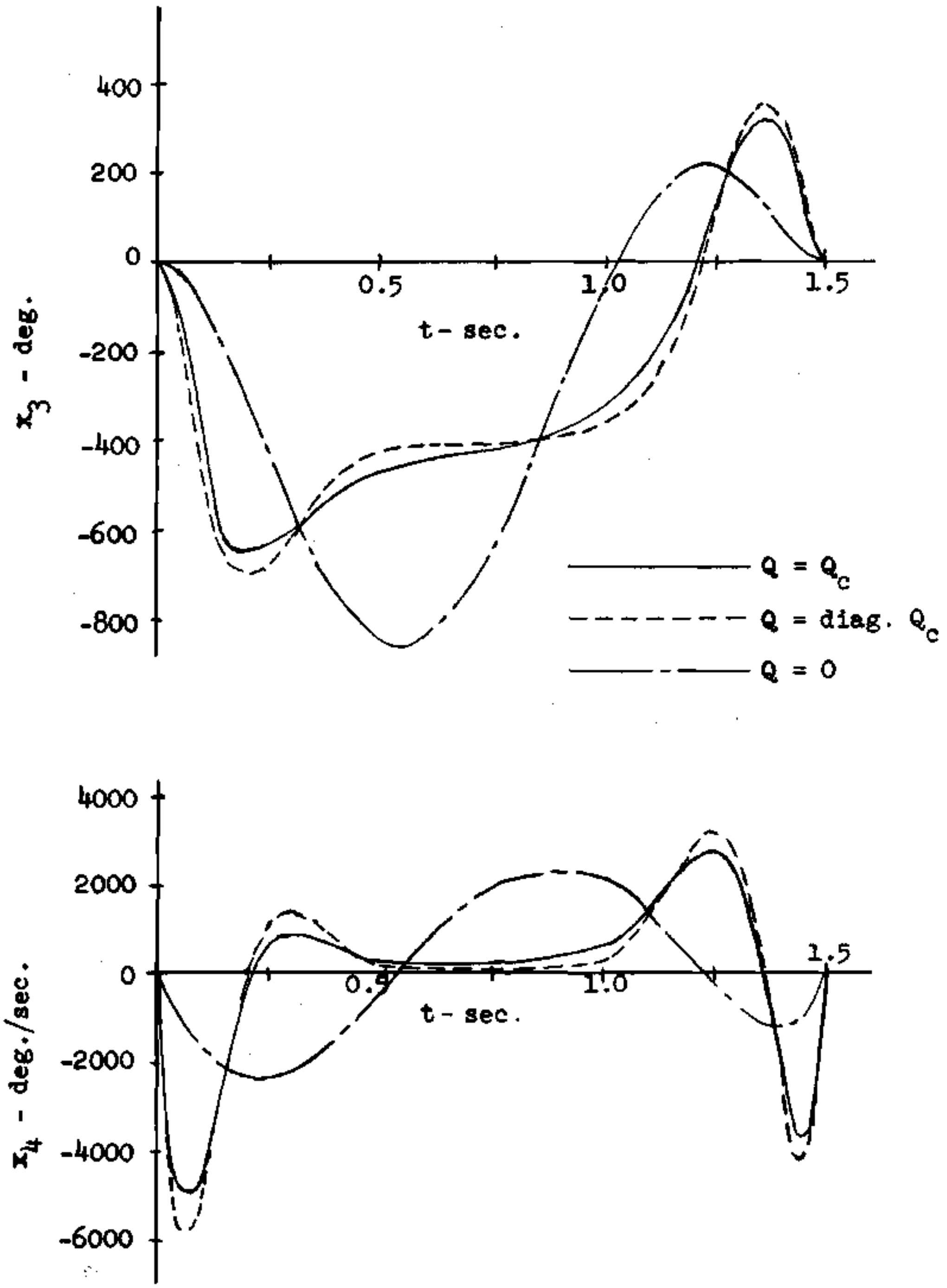

Fig. 24 Open-Loop Q Comparisons (x_3,x_4) For T = 1.5

selection) for the very short response interval problem T = 1.5 sec. The off diagonal terms are seen to be relatively unimportant in the present case. Furthermore, the simple case Q = 0 gives a smoother motion of the pilot motor and so might often be considered the most desirable choice of the three.

For response intervals longer than approximately T = 1.5 an added incentive is found for the weight selection Q = 0 in this problem. This is because the co-state sensitivities for the other cases become too large to allow use of single precision arithmetic on the longer response solutions. Thus, in this problem the weight selection Q = 0 is simple, gives "desirable" responses, is sufficient for linear optimality and offers a distinct computational advantage. The co-state sensitivity problem is illustrated in more detail later on.

Trajectory variations for differing response intervals are shown for the Q = 0 weight selection in Figures 25 and 26. In all cases the assumption of negligible pilot motor damping (D_1 = 0) was required to allow use of single precision arithmetic. Under these conditions a practical limit of T $\leq$ 6 sec. was found to result in workable co-state sensitivities. These plots are helpful in choosing a desirable response interval T.

Because of the co-state sensitivity problem the case of free time T was found to be beyond the limits of single precision arithmetic.[1] This is easily guessed for the weight selection Q = Q_c, since solutions beyond T = 1.5 are unattainable. The free time case for Q = 0 is equally difficult. In the latter case emphasis apparently

[1]Except for very small initial conditions x_o.

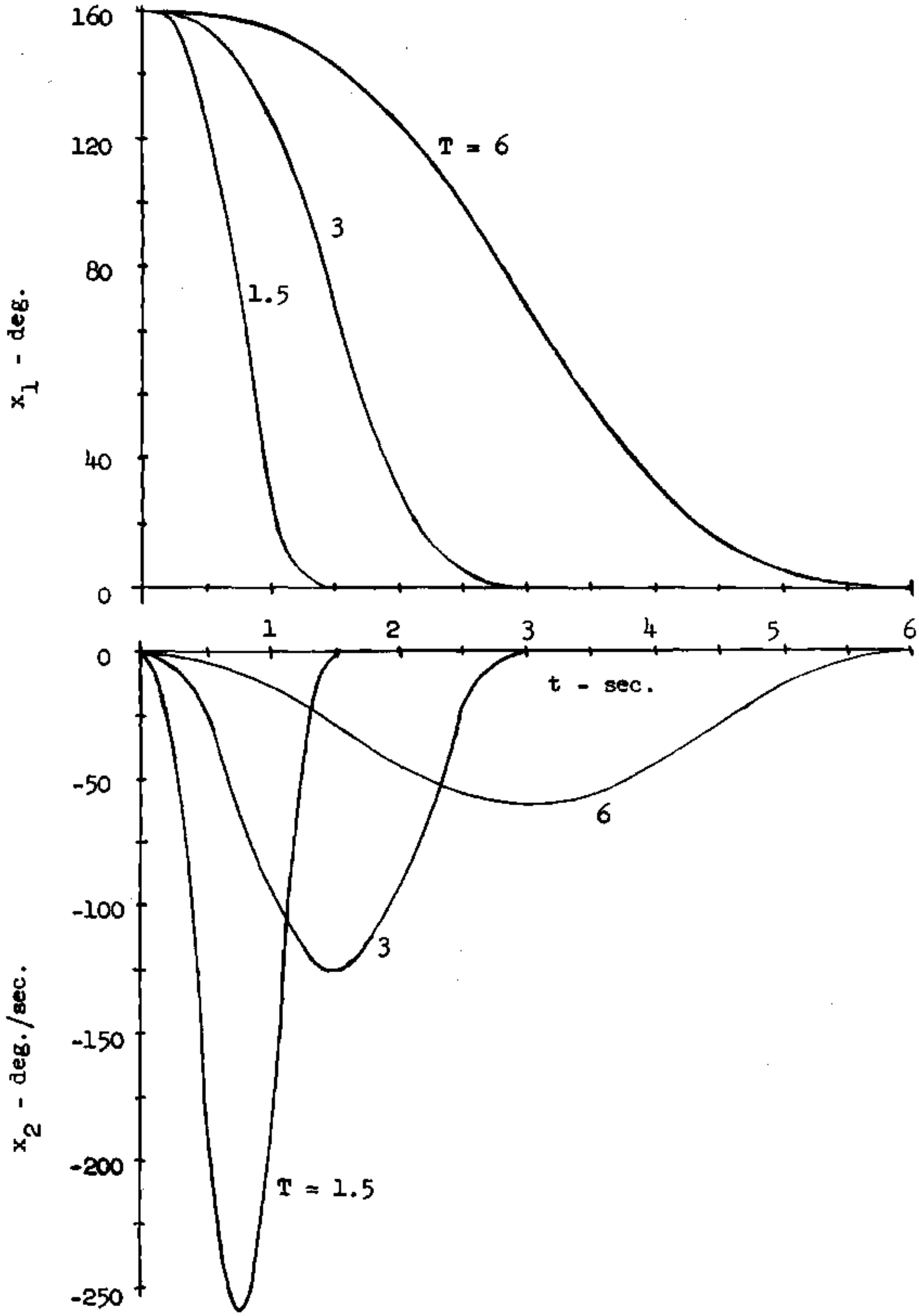

Fig. 25 Open-Loop T Comparisons (x_1,x_2) For Q = 0

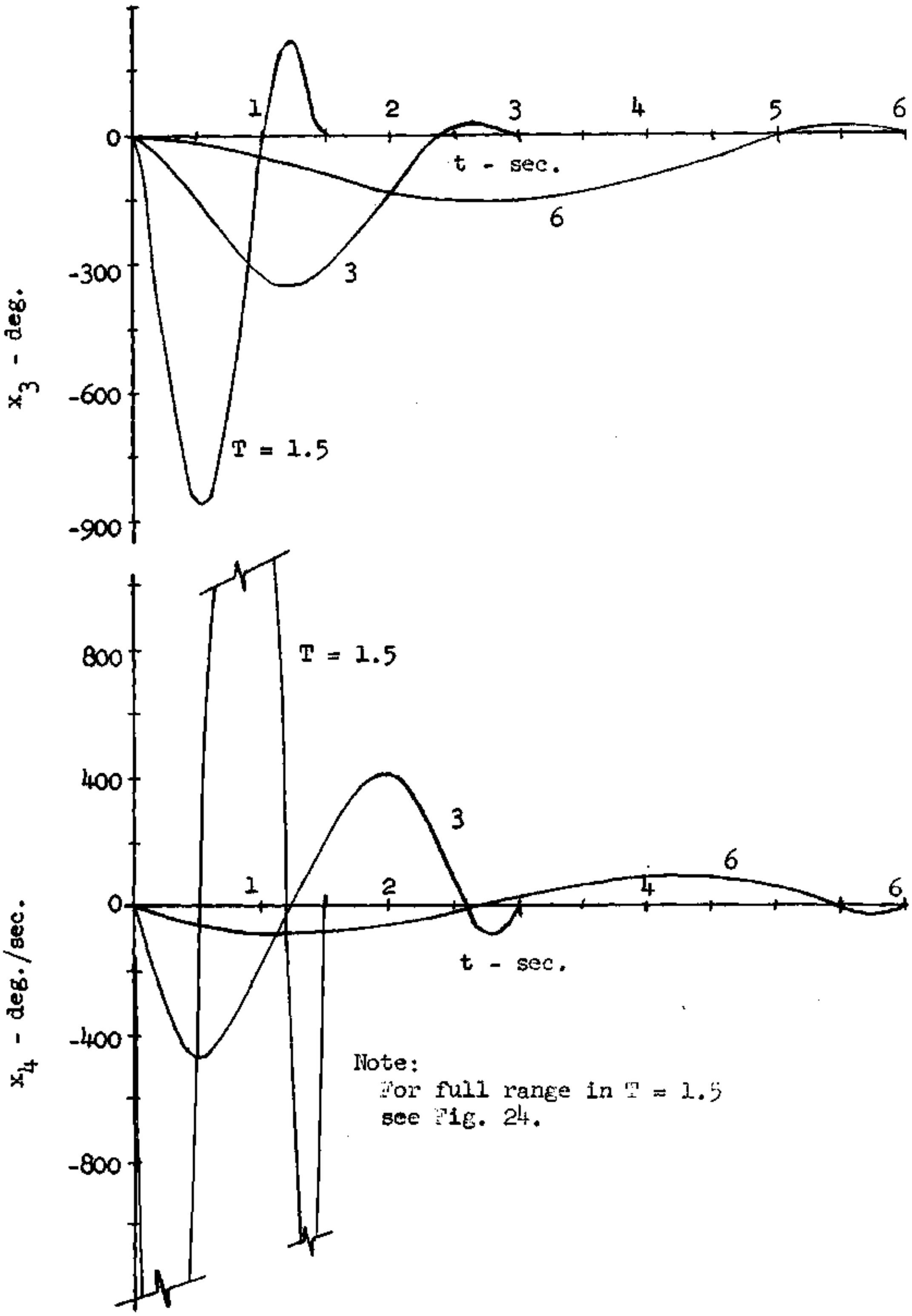

Fig. 26 Open-Loop T Comparisons (x_3,x_4) For Q = 0

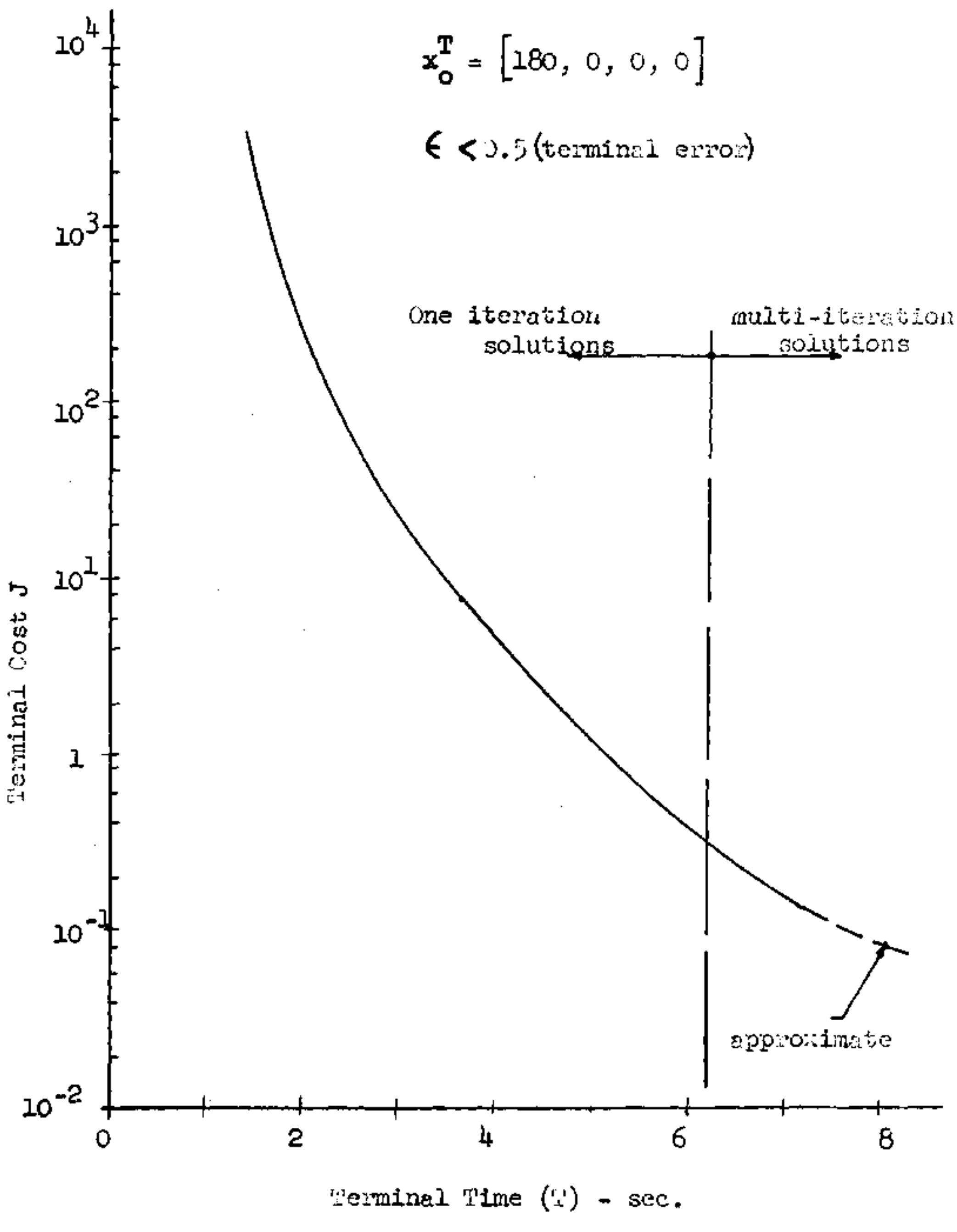

Fig. 27 Cost J Versus Response Time T (Q = 0)

goes into spreading out a low level control signal over a longer period of time, since no cost is assessed for nonzero state magnitudes. Figure 27 illustrates the cost versus time relationship in the Q = 0 case, based on a series of fixed time interval runs. It is seen that a cost minimum with the respect to T does not occur within the range of single precision calculations.

The flooding approach was no better than the adjoint method for solving the free time problem, for the same reason. Even the use of fixed time solutions by the adjoint method, to yield first guesses for the flooding variables, were found to be inadequate. Since the flooding condition (C-11) must be imposed, the fixed time (adjoint) solutions cannot be used directly. Moreover, any required small variations cause the free time flooding solutions to deviate wildly from their fixed time counterparts. While every flooding trajectory corresponds to some boundary value solution, and can even be stopped when a desired magnitude on any one state variable is reached, it must be appreciated that the remaining state magnitudes and time must simultaneously fall into some usable region of the state and time space as well. The experience of all attempts yielded no such "lucky shots."

(iii) Co-State Sensitivities

The extreme co-state sensitivity of the optimal trajectories in this problem has been mentioned. The adjoint variables provide an excellent means for examining the sensitivity problem in a quantitative manner. As discussed in Chapter II, the adjoint variables form a first order sensitivity matrix S(T) relating variations in the terminal state $\delta x(T)$ to small perturbations in the co-state initial conditions,

$$\delta x(T) = S(T)\, \delta p(0)$$

$$S(T) = [s_{ij}(T)],\ s_{ij}(T) = \frac{\partial x_i(T)}{\partial p_j(0)}$$

Table 3 documents the complete 4 x 4 fixed time sensitivity matrices computed with a linearized plant, for two cases of interest. For T = 3, the $Q = Q_c$, $D_1 \neq 0$ combination is a worst case, while the $Q = D_1 = 0$ case shows the total sensitivity reduction achieved by the latter assumptions. Figure 28 shows typical time variations in the sensitivities through plots of the main diagonal elements for the case $Q = Q_c$, $D_1 = 0$. Similarly, Figure 29 shows the time variation of the main diagonal elements for the least sensitive case $Q = D_1 = 0$.

Using the sensitivity figures in an order of magnitude analysis, limits on the response interval for various cases can be predicted for single precision calculations. Very roughly, when the sensitivities exceed the range 10^{12} - 10^{14}, single precision corrections (twelve significant digits) begin to fail (depending on the order of magnitude of the last co-state vector, etc.). No double precision calculations were attempted, partly because the Burroughs machines are not set up to encourage this.[1]

The sensitivity problem also reveals itself in the flooding approach. It turns out that the adjoint boundary value solutions cannot later be duplicated by flooding unless the full twelve place accuracy of the adjoint solutions are carried in the flood starting vector. This includes the exact value of the nonzero state vector x(T) (which is only reduced to within a small ε circle by the adjoint

[1]Extensive reprogramming, using an obscure Polish notational system, is required of all relevant calculations and input/output processing.

Table 3. Co-State Sensitivity Matrices ($T = 3$)

$Q = Q_c$, $D_1 \neq 0$

-9.386×10^{19}	1.742×10^{21}	-1.976×10^{22}	3.667×10^{23}
-1.742×10^{21}	3.234×10^{22}	-3.667×10^{23}	6.807×10^{24}
-2.836×10^{22}	5.264×10^{23}	-5.969×10^{24}	1.108×10^{26}
-5.264×10^{23}	9.771×10^{24}	-1.108×10^{26}	2.057×10^{27}

$Q = 0$, $D_1 = 0$

-1.441×10^{5}	5.049×10^{5}	-3.526×10^{4}	5.077×10^{4}
-5.049×10^{5}	1.724×10^{6}	-5.077×10^{4}	5.523×10^{4}
-2.529×10^{6}	8.563×10^{6}	-1.665×10^{5}	1.665×10^{5}
-8.563×10^{6}	2.865×10^{7}	-1.665×10^{5}	1.110×10^{5}

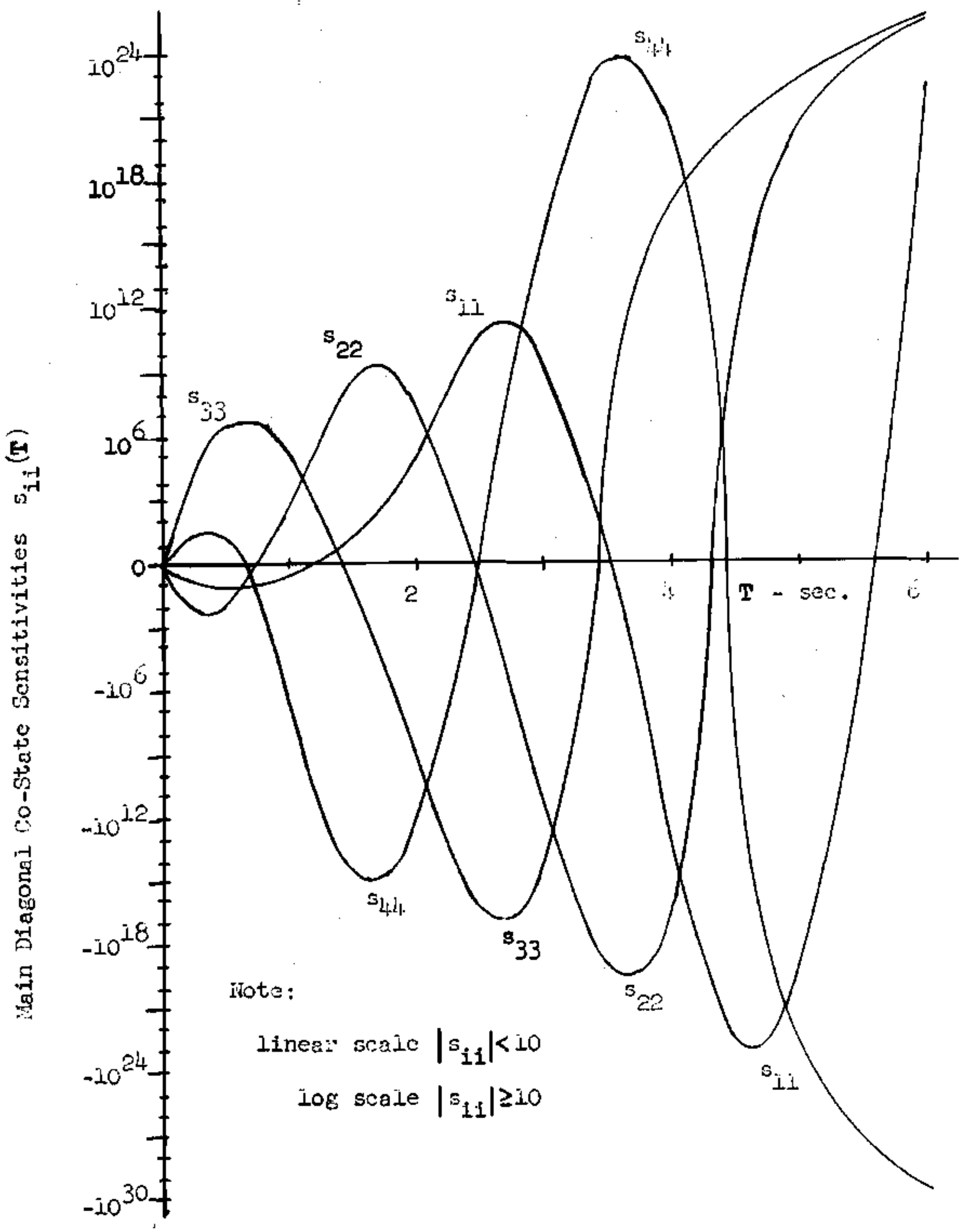

Fig. 28 Time Varying Co-State Sensitivities ($Q = Q_c, D_1 = 0$)

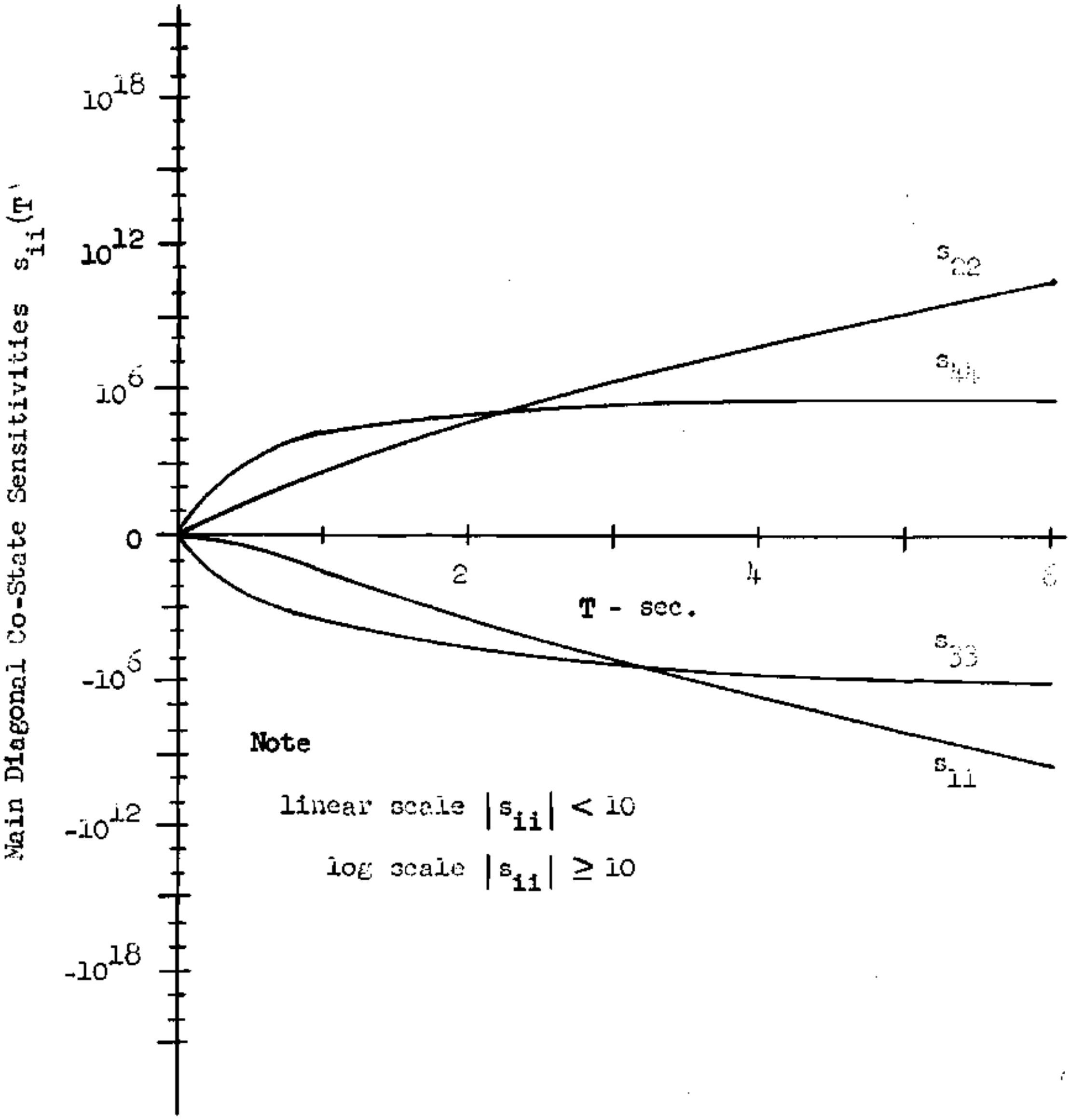

Fig. 29 Time Varying Co-State Sensitivities ($Q = 0$, $D_i = 0$)

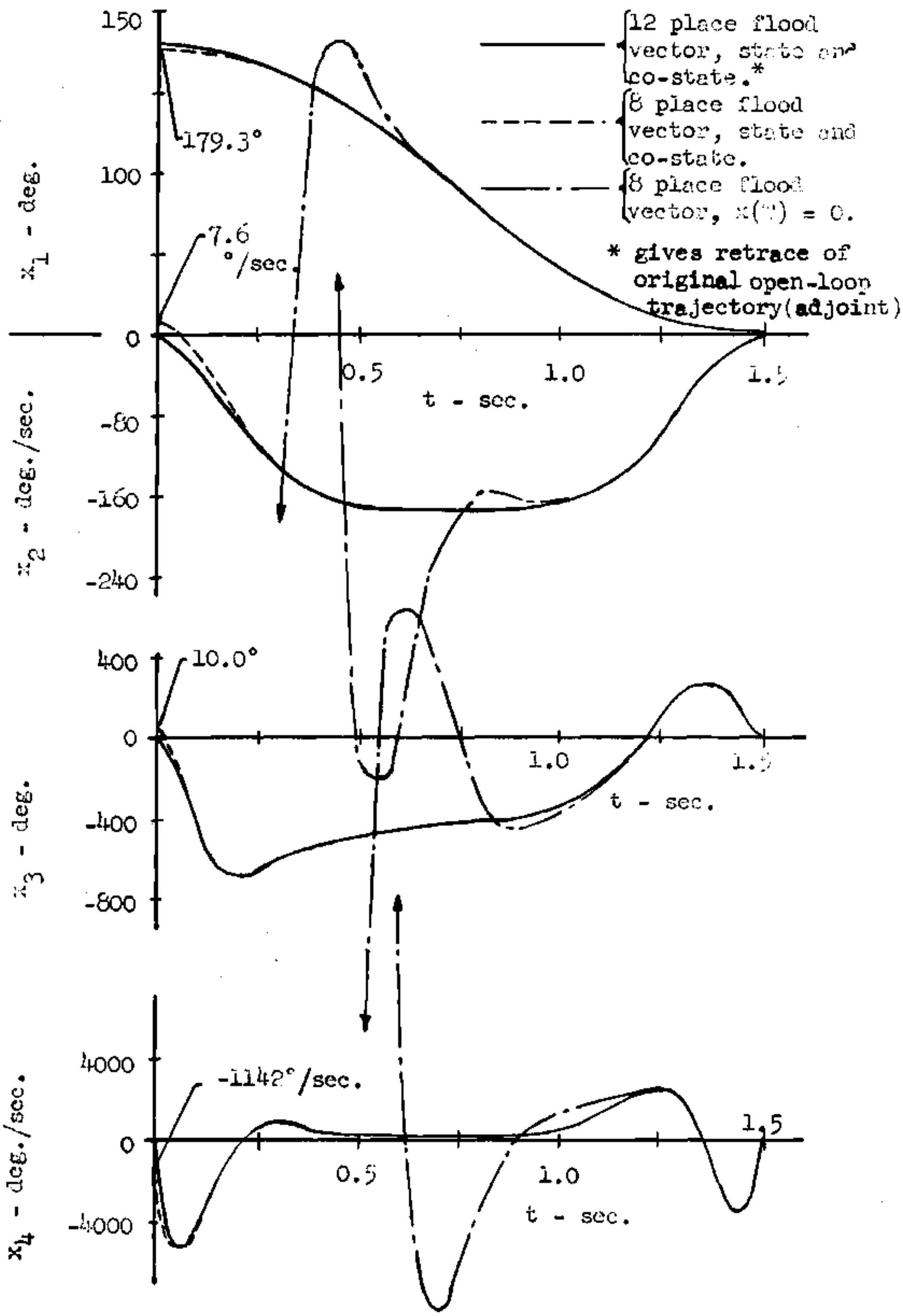

Fig. 30 Illustration Of Flooding Sensitivity

solutions). Figure 30 illustrates the problem.

(iv) Nonlinear Solutions

Thus far all open-loop solutions have been based on the linearized model of the nonlinear servomotors. Even in this simplified form, substantial computational restrictions on the choice of solutions (i.e., $T \leq 6$ for $Q = 0$ and $T \leq 1.5$ for $Q = Q_c$) are observed due to the limitations of single precision arithmetic imposed by the high co-state sensitivities. With the nonlinear model, a further serious computational difficulty is found, the problem of limited convergence rates.

When the initial conditions x_o are sufficiently small, nonlinear boundary value solutions are found by the adjoint method in a reasonable number of iterations. Figure 31 illustrates a typical nonlinear response from the "small condition" region. Starting with a 20° offset on the output shaft, the solution as shown was found in three iterations and four sweeps[1] with a total calculation time of 5.12 minutes.[2] The corresponding linearized response is also shown for comparison.

When the 20° offset is moved to 40°, the nonlinearities take a heavy toll in convergence rate. Figure 32 shows the iteration history of the attempt, which was terminated at 10 iterations and 22 sweeps in 16.85 minutes. The reason for the slow convergence is seen to be caused by the repeated need for a small correction step size ($c_{ii} = 0.1$,

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[1]An iteration is counted every time the rms boundary value error is reduced over the value from the previous iteration. A sweep is counted every time an attempt is made to reduce the error. The difference between sweeps and iterations is therefore the number of failures to reduce the error. A failure causes a correction step size reduction. An iteration causes a step size increase toward the maximum limits.

[2]All times are for Burroughs B-5500 installation at Georgia Tech.

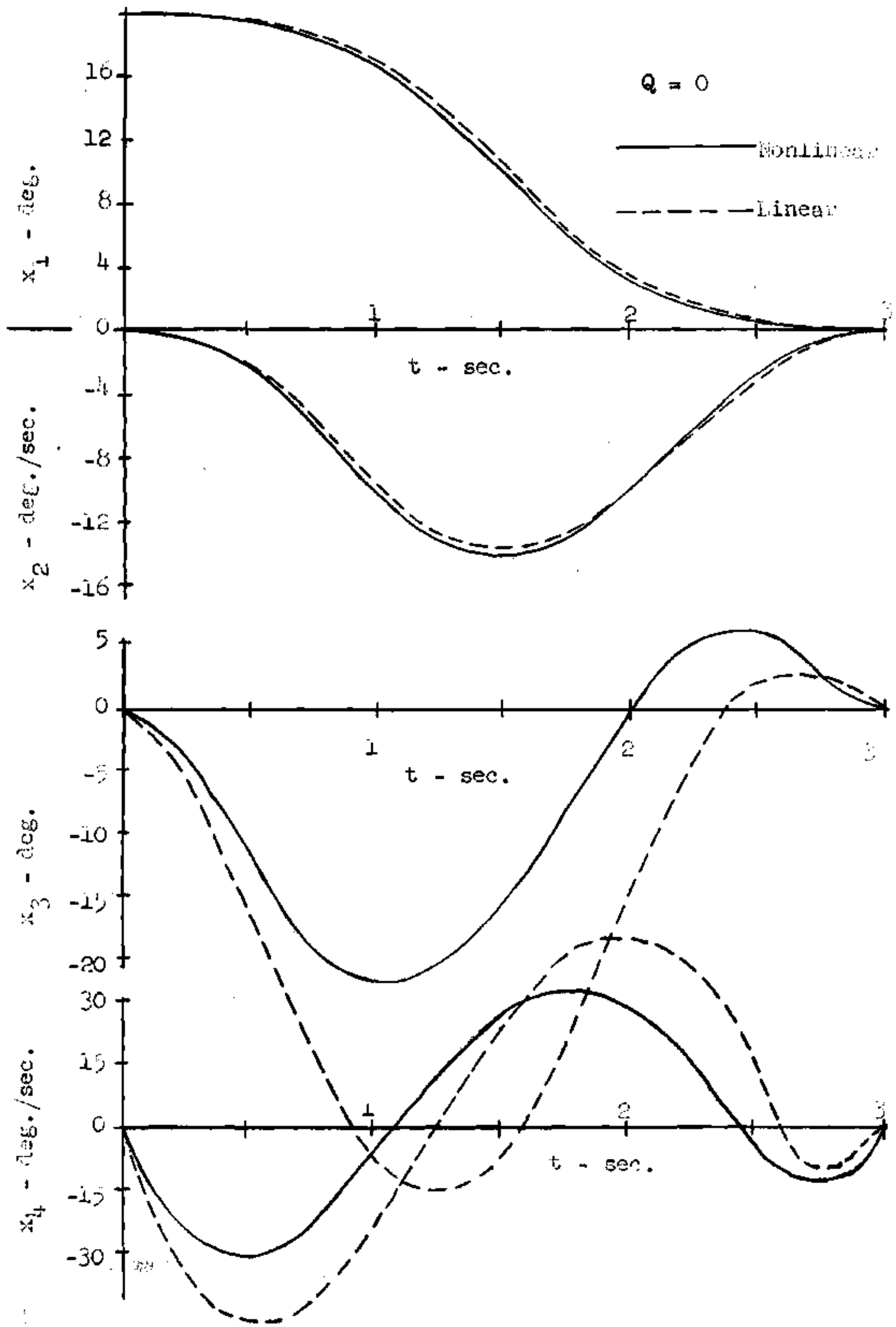

Fig. 31 Nonlinear Open-Loop Response

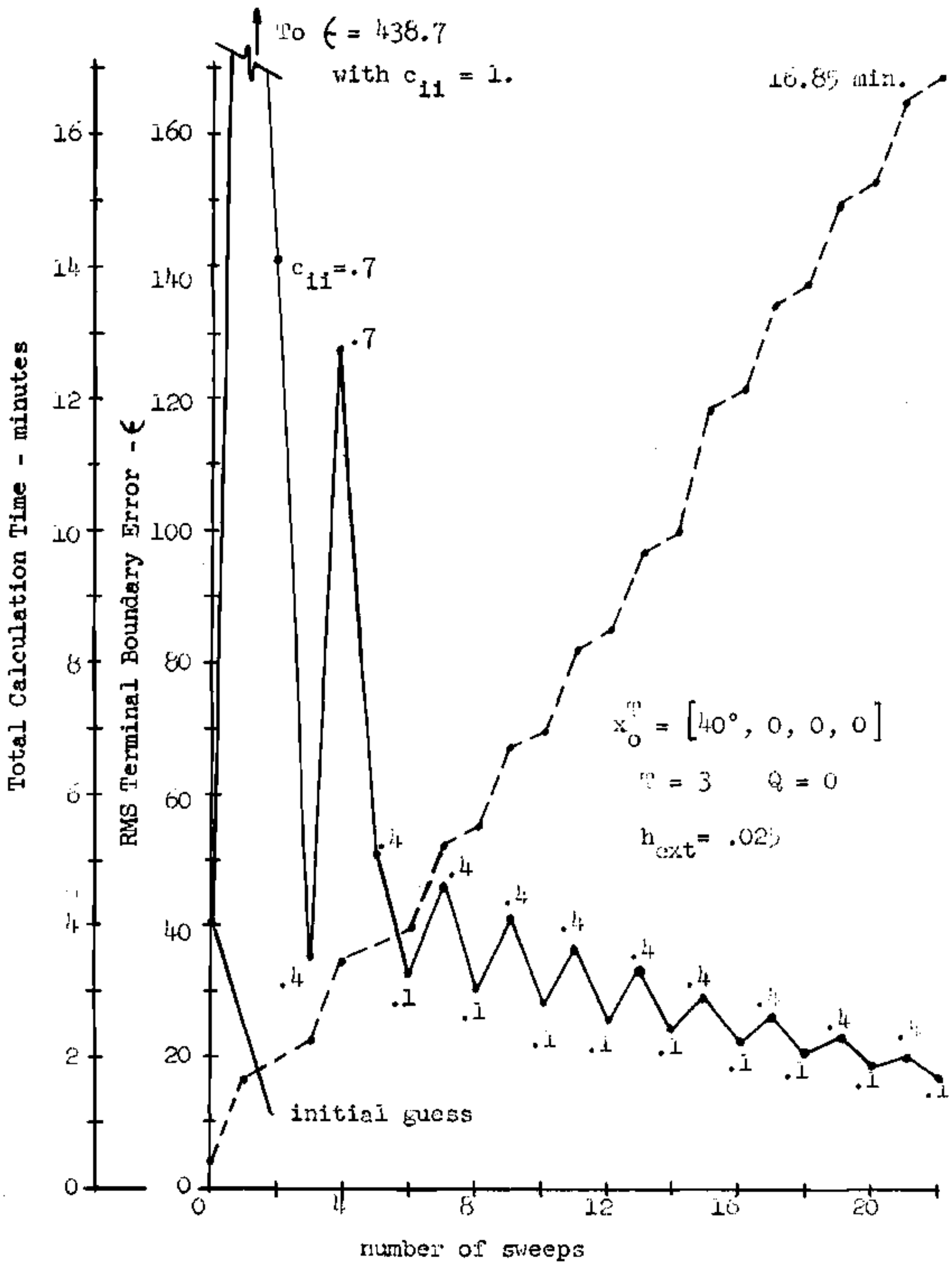

Fig. 32 A Nonlinear Iteration History

equation (2.15)). Based on the experience of the successful 20° offset solution, a faster convergence rate would not be found until the boundary value error was reduced to approximately 10. On this basis it can be estimated that the 40° offset problem would require roughly 1/2 hour to solve.

Even the 40° offset starting condition lies far short of the limits needed to cover a practical application of this problem (see Table 4 for list of starting points on all four state variables used in the data base). Thus, even when the sensitivity restrictions are observed, the nonlinear solutions of this problem would require well in excess of 1/2 hour per open-loop trajectory. Some time reduction might be achieved by attempting to use the adjoint sensitivity matrix on more than one iteration, in regions where small correction step sizes are needed. A dead zone effect could be included in the correction step size control loop to inhibit excessive hunting during periods of slow change in the convergence rate (see Figure 32). Unbalanced step size weights could be tried (i.e., $c_{ii} \neq c_{jj}$). These and other ideas might be interesting to pursue. However, in view of the other difficulties associated with this problem, no further time was expended toward generating the nonlinear solutions. It is remarked that the flooding approach was not attempted because of the demonstrated sensitivity problem associated with it on the linearized model.

(v) Closed-loop Control Functions

A limited number of least squares control function fittings were attempted from the computable linear model open-loop optimal solutions. Free time (T) responses are therefore not considered.

However, both short (T = 1.5) and moderately long (T = 6) interval problems are included.

Three least squares control functions are considered. Two time varying gains (TVG) models are analyzed. The first of these is the sixteen term model

$$\hat{u} = -(K\tau)^T x : \ \text{TVG}_{16} \text{ model} \tag{4.9}$$

$$\tau^T = [1,\ t_g,\ t_g^2,\ t_g^3]$$

$$K = [k_{ij}] \quad i,j = 1,2,3,4$$

This model is extended to nineteen terms in the second TVG case by adding further degrees of freedom to the x_4 gain. The selected form is

$$\hat{u} = -(K\tau)^T x - (k_{17} + k_{18}t_g + k_{19}t_g^2)t_g^3 x_4 : \text{TVG}_{19} \text{ model} \tag{4.10}$$

In the third case a stationary linear feedback law is assumed, i.e.,

$$\hat{u} = -k^T x \quad : \ \text{linear model} \tag{4.11}$$

$$k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}$$

Two open-loop data base tapes were written based on a set of eight standard initial conditions (x_o). Table 4 lists the eight vector

Table 4. Data Base Initial Conditions

No.	x_1 deg.	x_2 deg./sec.	x_3 deg.	x_4 deg./sec.
1	180	0	0	0
2	180	120	0	0
3	180	120	1800	0
4	180	120	1800	360
5	180	0	0	360
6	0	120	1800	0
7	0	120	1800	360
8	0	0	1800	360
9	180	120	0	360
10	180	0	1800	360

points. For the T = 1.5 problem a 1208 point data set was written using the conditions $Q = Q_c$ (weighting matrix 4.8 derived from Bell's approach) and $D_1 = 0$ (negligible pilot motor damping). Figures 23, 24 show time histories from initial condition one of this set. For the T = 6 problem a 960 point data set was written for the case Q = 0, $D_1 = 0$. Figures 25, 26 show time histories from initial condition one of this set. Typical closed-loop simulation results for the short time TVG_{16} (4.9) and TVG_{19} (4.10) models are shown in Figures 33 and 34. Both responses follow the open-loop at first, but eventually deteriorate. None of the two functions was satisfactory over the full response interval. All eight Table 3 trajectories were tested with each control. The added terms associated with the x_4 gain in the TVG_{19} model are seen to give a definite improvement over the 16 term case. It is very likely that further experimentation with the basis function and weighting of the data base would yield further improvement. However, even a sixteen term function could hardly be regarded as a desirable control law. The poor showing of these high order functions is a noteable discouragement.

A typical closed-loop simulation for the moderately long T = 6 problem is shown in Figures 35 and 36. Results are given for the linear (4.11) control. Some improvement could probably be made with further experimentation. However, a single attempt with a TVG_{16} control gave a diverging response.

In view of the nonzero controls at t = T (Figs. 34, 35), a constant term in the control function (as in Case One) might be of some benefit.

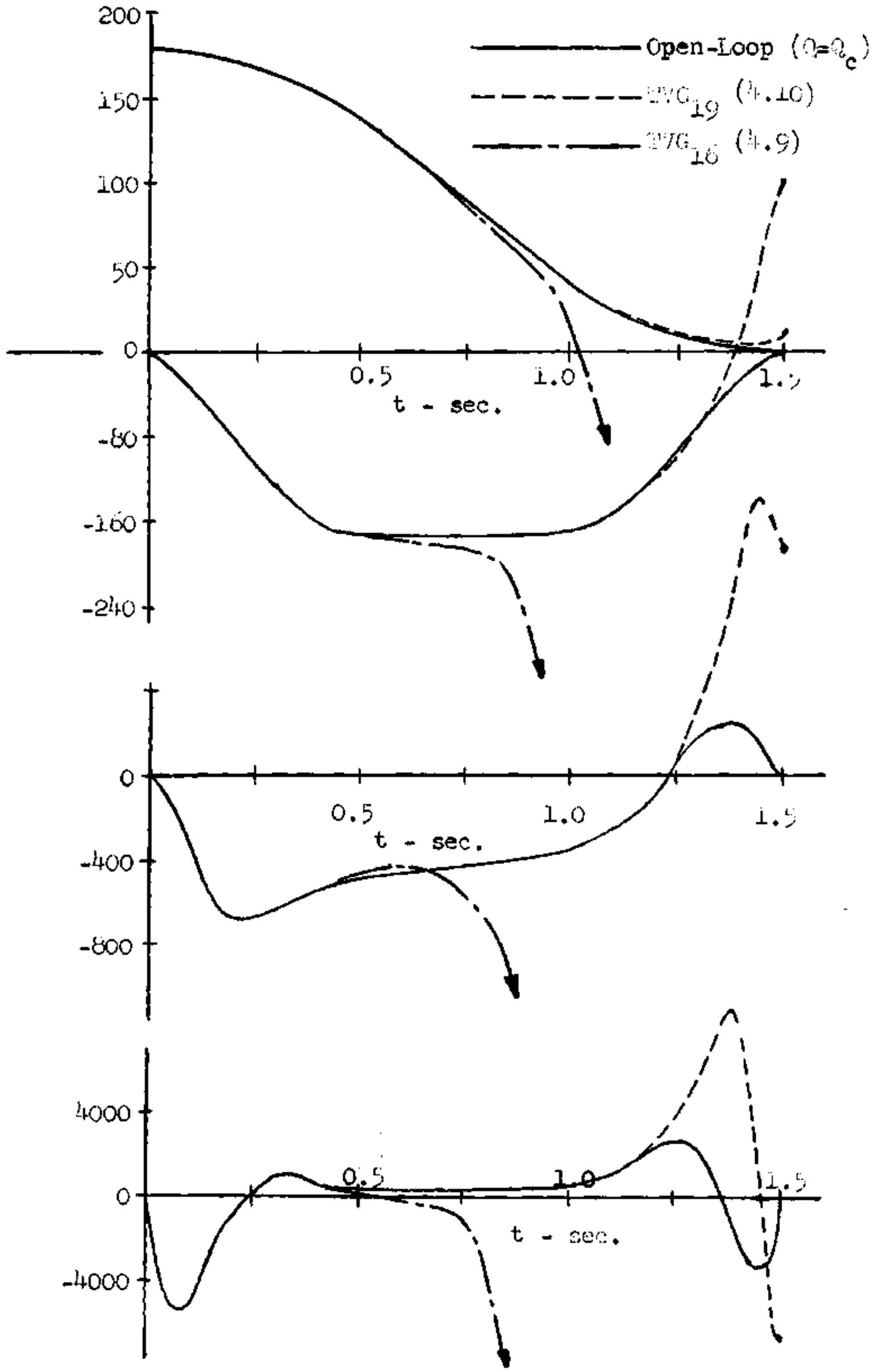

Fig. 33 Closed-Loop State Histories, T = 1.5

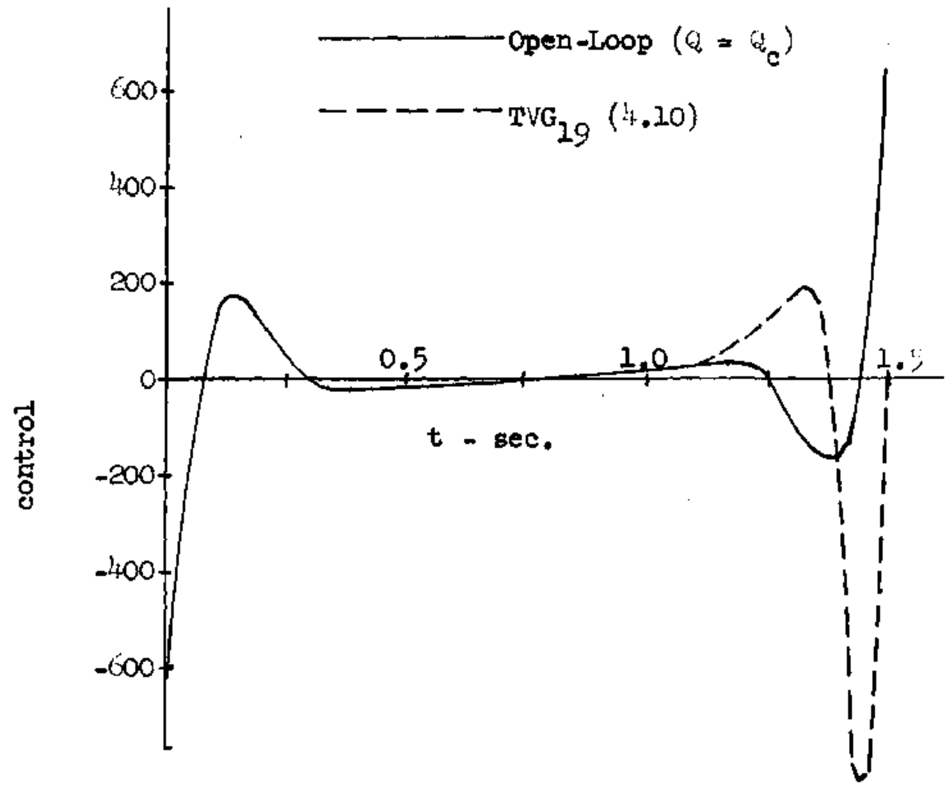

Fig. 34 Closed-Loop Control Histories, T = 1.5

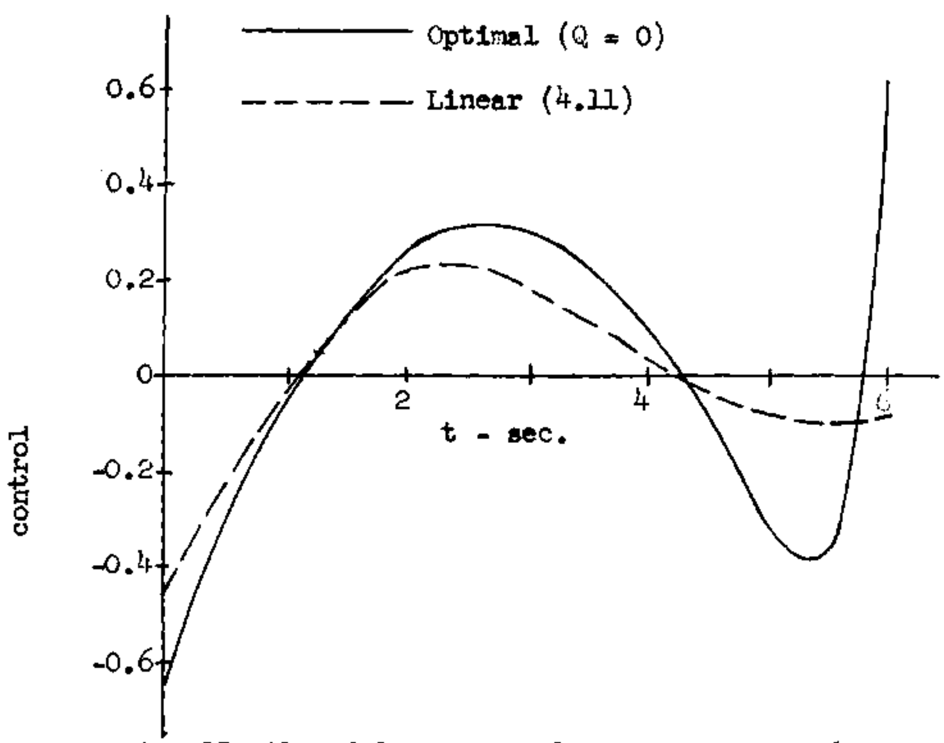

Fig. 35 Closed-Loop Control Histories, T = 6

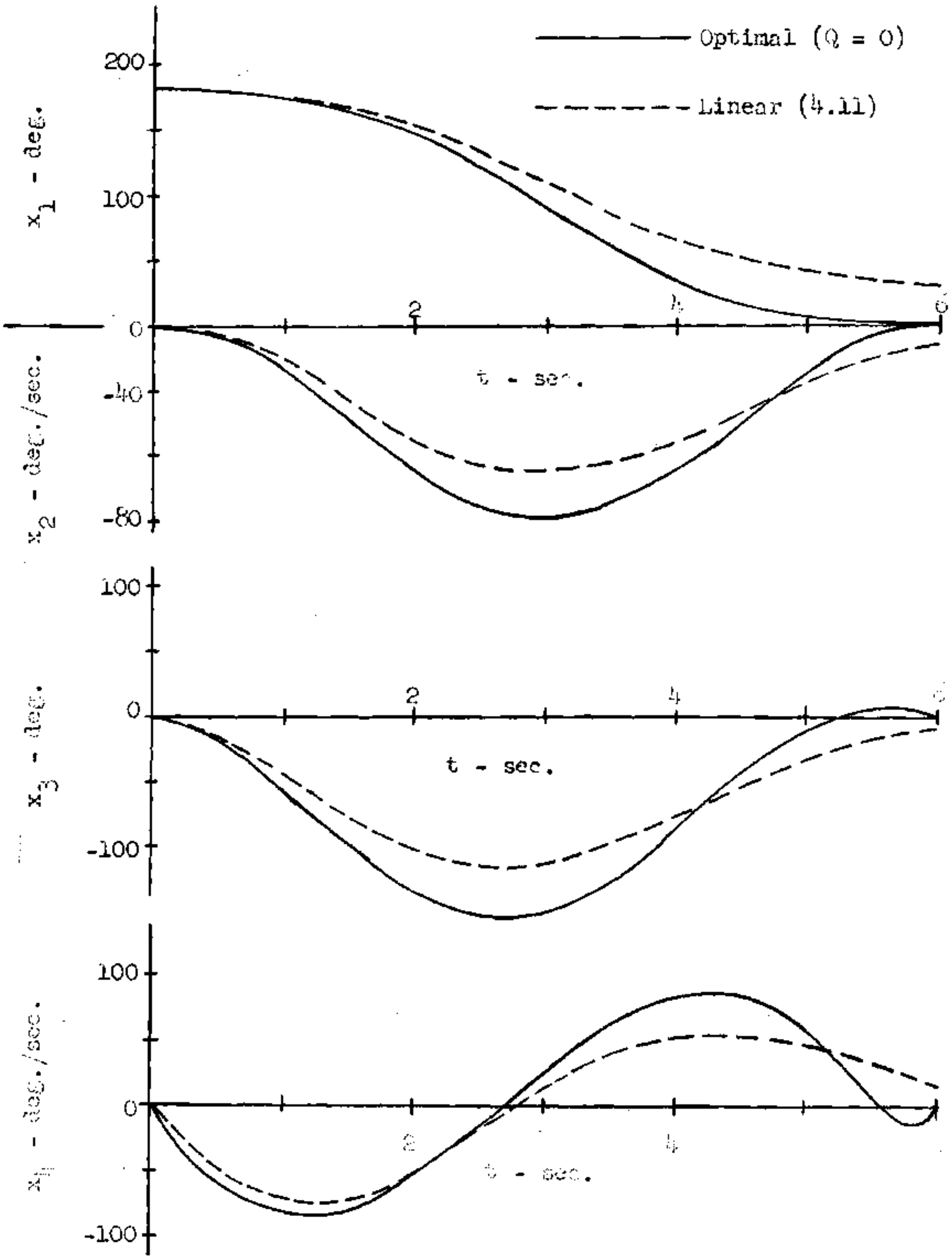

Fig. 30 Closed-Loop State Histories, T = 6

(vi) Calculation Costs

Calculation costs have already been discussed for the nonlinear model, and were shown to be prohibitive in the present case without the development of some means to speed the convergence rate.

When calculation noise considerations permit, the linear open-loop solutions are obtained in one iteration. In this case a rough rule-of-thumb for computation time in this problem is 0.01 minutes per solution per data point. This is based on a printout ratio of one in every 10 data points. In the case of one iteration solutions the printout time plays a more dominant role in the total than for the multi-iteration type. Thus, if every data point is printed on a single iteration solution, the above figure increases to about 0.03 minutes per point. If integration step size requirements call for 120 data point calculations, and all 120 points per trajectory are printed and stored on magnetic tape for the data base, then an eight trajectory data base (960 points) requires about 29 minutes of computer time to write. This figure is in good agreement with actual experience.

Similarly, the program used to calculate the least squares control function charges roughly 0.065 seconds per term per data base point. Thus, a 16 term control function constructed on the 960 point data base costs about 1 1/2 minute. A simulation run as a closed-loop system then costs an added 0.21 seconds per data point per trajectory, when all data points are printed. To compare each control function with the original eight trajectory data base, plus a few extra not on the base (say 10 total), the simulation cost comes to about 4.2 minutes per control function. A study of five control function fittings on the original data base might therefore typically cost another 29 minutes, giving a total of close to one hour of B5500 computer time for the complete problem.

The above time estimates are based on straight tare free calculations of one iteration linear solutions. Further remarks on costs are left to the discussion section below.

(vii) Discussion

Some of the more important points of this case study are now summarized. Regarding the system under study, the double integrations in the plant make the system inherently difficult to stabilize.

The weight selection method of Bell was given special attention. In the computable short T case the corresponding responses could likely be suitable for many applications (Figures 23 and 24). It was found that the off diagonal terms were of small importance in shaping the response. A very simple alternate choice Q = 0 was also tried. This response was an improvement in the sense that the pilot motor motion was smoother (Figure 24). Furthermore, the co-state sensitivity problem was less critical in the latter case, allowing longer time response solutions to be made (Figures 25 and 26).

The co-state sensitivity problem of this case study was observed to be acute. Table 3 documents the high order of magnitudes typically found. Because of the sensitivity problem, solutions could not be computed with single precision arithmetic without the added assumption of negligible pilot motor damping ($D_1 = 0$). Even then, computable response intervals were limited to not much over T = 6 sec. Free time solutions were not possible (Figure 27). Figures 28 and 29 illustrate the rapid growth of the sensitivities with time. Flooding was essentially impossible without knowing the answer in advance to 12 significant figures (Figure 30). Conversion to double precision arithmetic would reduce the sensitivity problem and allow a wider range of solutions. Double precision conversions are not conveniently implemented on the Burroughs machine, though it can be done if the added effort and calculation time is warranted.

The nature of the nonlinear functions of this problem was found to cause excessively slow convergence rates except when starting from fairly small initial conditions. A small region nonlinear trajectory is compared in Figure 31 with its linear system counterpart. The convergence problem is illustrated in Figure 32 by the iteration history for a moderate sized initial condition. Convergence was estimated to require about 30 minutes of computer time if the search had been allowed to continue. Much larger calculation times would be required for the large initial conditions of Table 4. Thus, fitting of control functions based on nonlinear open-loop responses was prohibitive without modifications to the adjoint approach. A few possible modifications were mentioned but no further attempts were tried. The flooding approach was of no help due to the flooding sensitivity problem.

Calculation time costs were discussed in relation to this problem. A rough breakdown was estimated at 1/2 hour for typical data base generation and one half hour for fitting and testing a set of 5 closed-loop control functions. However, this is a tare free estimate. Experience with this case shows that even after program de-bugging charges are added, a far greater amount of time can accumulate while various problem areas are identified, checked out, and possible alternatives assessed. In short, costs will very likely be high.

Regarding closed-loop control fittings, the computable open-loop control surfaces used in this study were apparently very complex over the wide range of data base initial conditions (Table 4). Much improvement could no doubt be obtained if operation was restricted to narrower data base sets, as was shown in the first case study. However, such operation might find little practical application. It is known from closed form solutions that as T becomes large the control surface approaches the linear stationary control function (4.11). In this case a well fitting control function would most probably be obtained. However, because of the co-state sensitivity problem the open-loop data base for $T > 6$ was unattainable with the single precision calculations.

It is emphasized that the values of the 11 basic constants which define the plant are, in a sense, not arbitrary. Rather, they were carefully derived in a way to yield a set of values typical of those to be found in practice (Appendix C). Thus, the nonpositive Q matrix derived by Bell's method, the high co-state sensitivities, the slow nonlinear convergence, and all other results of this section can be said to arise from a realistic design situation.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

A set of known design techniques have been assembled as a design procedure, and the problems of applying the procedure to a class of plants has been empirically examined. In those problems where it can be applied, the procedure leads the design process on a five step path to the solution of a feedback control function of the form $\hat{u} = \sum^{r} k_i Z_i(x,t)$.

The computer solutions show, in a meaningful qualitative sense, that typically satisfactory design solutions can be produced for plant cases within the considered class. However, a major restriction is that actual success appears to be highly problem dependent and difficult to predict a priori, as revealed by the third case study.

The experimental results are developed through the application of the procedure to three case studies. The purpose of the first case was to test the procedure on a problem with known form of solution. The results illustrate that least squares derived control functions are rather sensitive to a proper selection of the control function form. Thus, a time varying gain control form worked rather well (corresponding to the known form of solution) whereas a second order power series did not.

The step-up procedure was shown to be an effective means of selecting, on the basis of least squares, relatively efficient closed-loop

Z_i functions, as measured by simulation performance in the first case problem. This held true over a variety of data base selections, even when the available choices were not good. Thus, the least squares criterion appears to permit efficient discrimination between candidate Z_i selections for closed-loop controls. In more absolute terms, however, a high level of least squares correlation coefficient was found to be an insufficient basis for predicting adequate closed-loop performance. Thus, the need for the closed-loop simulation step was securely established.

The purpose of the second problem was to examine results among four subcases of cascade gain element shapes. The free response time problem was considered. As the gain shape nonlinearity was increased, closed-loop simulation results became more critically effected by the data base selections. Thus, the importance of data point selection was revealed as a means of improving the least squares fittings. Results again varied with the Z_i selections. However, for each subcase of gain shape a least squares control function was found which gave favorable comparisons with the open-loop response.

In the second case study the weight selection Q was positive definite, thus assuring sufficient conditions of optimality of the open-loop solutions. However, all closed-loop simulations produced essentially the same cost figure J. Thus, the performance index, while effective as a means of shaping the trajectory responses, appears to be rather insensitive as a performance figure of merit. On this basis the importance of optimality appears doubtful and the true significance of the quadratic index as a response shaping device is emphasized.

The third case problem consists of a fourth order plant with nonlinear two phase servomotors. The purpose was to consider a more complex higher order case. The results, however, reveal problem dependent difficulties in the application of the method. Thus, without double precision arithmetic, extreme co-state sensitivities limited response intervals to moderate lengths. Slow boundary value convergence for the nonlinear plant model made computational costs with the adjoint method prohibitive. Moreover, the flooding technique could not be used as an alternate method because of the high co-state sensitivities.

The high sensitivities could be predicted by examining the 2nx2n transition matrix for the Pontryagin equations, suitably linearized. However, when the plant is of third order or higher it is perhaps easier to program a trial solution as a test. Long response time problems are more susceptible to the difficulty. Reducing the amplitude weights Q to zero was found to give substantial relief. Thus, a problem calling for heavy weights, which show up as coefficients in the differential equations, is more vulnerable to the co-state sensitivity problem. Double precision calculations, at added cost, may resolve the trouble.

Slow boundary value convergence rates also appear difficult to predict in advance. A trial test is perhaps the quickest way to resolve the question. The adjoint method worked quite well when the initial conditions were selected sufficiently close to the origin (e.g., 20° of position offset gave rapid convergence in the third problem, but 40° was estimated to require one half hour of computer time per trajectory). Linearization of the plant equations, when permissible, allows one step convergence.

Bell's weight selection method usually gave a reasonable balance of values in the finite time and nonlinear applications, especially as a first estimate. More often than not, the method seems to give non-positive Q based on standard root location selection criteria. In the first case study a negative Q element was observed to give positive rate feedback, allowing a less sluggish response. With regard to finding acceptable degrees of relative stability, the weight selection problem using Bell's method and tables, did not appear to be critical in the examples covered. In view of the successful results with nonpositive Q, the question of optimality with the quadratic index appears to be largely incidental to the engineering problem.

The linear plant examples of the case studies have been useful in examining the design procedure. However, they are less significant as practical applications since use of the quadratic index with linear plants admits a more readily obtained exact solution through the integration of a matrix Riccati equation. Thus, when problem dependent difficulties with the boundary value solution permit, the nonlinear plant offers the best potential area of application for the present method.

Concrete examples of this are demonstrated with the nonlinear plants of Case Two, when applied to the feedback functions with four or more gain constants. Thus, the two gain constants for the linear feedback law could more easily be found by experimental adjustment on an analog computer. However, with the four and eight gain functions (which gave improved performance) a feasible attempt at trial and error adjustments is less easily visualized. Even then, the result applies to only one specific selection of Z_i functions. By contrast, once the open-loop data has been stored on tape, least squares gain calculations can be rapidly made for several control models. Moreover, the rapidity of the least squares calculations combined with the discriminating ability of the criterion makes possible a search and selection of efficient Z_i combinations from a large pool of terms by the step-up procedure. Note that the search efficiency advantage of the step-up approach could not be realized by any competing procedure lacking these two qualities.

It is recommended that any further work in this area could best profit from attempts to relate the control function fitting more directly to the cost criterion and the terminal boundary conditions. Emphasis could perhaps best be placed on applications to true optimization problems, i.e., where the cost criterion is an explicit design requirement which fully justifies the effort expended in seeking the boundary value solutions, and where any required complexity of the feedback hardware would be an implied necessity.

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APPENDIX A

CASE ONE FORMULATION

The plant considered for Case One is governed by the two state variable differential equations:

$$\dot{x}_1 = x_2 \tag{A-1}$$

$$\dot{x}_2 = -\omega x_2 + Ku$$

The cost index is taken to be

$$J[u] = \int_0^T (q_1x_1^2 + q_2x_2^2 + u^2)dt \tag{A-2}$$

where T is a fixed response time interval and all non-diagonal terms in the weighting matrix Q have been set to zero.

The Hamiltonian function is

$$H = -(q_1x_1^2 + q_2x_2^2 + u^2) + x_3x_2 + x_4(-\omega x_2 + Ku) \tag{A-3}$$

where x_3 , x_4 are the co-state variables. From (A-3) the Maximum Principle yields

$$u^* = \frac{K}{2}x_4 \tag{A-4}$$

Based on (A-3) and (A-4), the following four first order differential equations comprise the necessary conditions:

$$\dot{x}_1 = x_2 \tag{A-5}$$

$$\dot{x}_2 = -\omega x_2 + \frac{K^2}{2} x_4$$

$$\dot{x}_3 = 2q_1 x_1$$

$$\dot{x}_4 = 2q_2 x_2 - x_3 + \omega x_4$$

The appropriate two point boundary conditions for (A-5) are

$$x_1(0) = x_{10} \qquad x_1(T) = 0 \qquad T = \text{fixed} \tag{A-6}$$

$$x_2(0) = x_{20} \qquad x_2(T) = 0$$

The adjoint differential equations required by the general computer program for the numerical solution of (A-5), (A-6) are

$$\dot{\lambda}_1 = -2q_1 \lambda_3 \tag{A-7}$$

$$\dot{\lambda}_2 = -\lambda_1 + \omega\lambda_2 - 2q_2\lambda_4$$

$$\dot{\lambda}_3 = \lambda_4$$

$$\dot{\lambda}_4 = \frac{-K^2}{2} \lambda_2 - \omega\lambda_4$$

The boundary values are supplied automatically by the general integration program, as discussed in Chapter II. Since the equations

(A-5), (A-7) are linear, only one iteration is required for any reasonable starting guess of the co-state initial conditions $x_3(0)$, $x_4(0)$ and specified accurancy tolerance on the boundary conditions.

The plant constants K, ω were chosen for comparative purposes to correspond to the plant considered by Bell[1], i.e.,

$$K = 1.0$$

$$\omega = 1.0$$

[1]Ref. 1, pp. 45.

APPENDIX B

CASE TWO FORMULATION

The plant for Case Two is governed by the nonlinear state differential equations:

$$\dot{x}_1 = x_2 \tag{B-1}$$

$$\dot{x}_2 = -\omega x_2 + KF(N,u)$$

$$F(N,u) = \mathrm{sgn}(u)M\left|\frac{2b}{N}u\right|^{\frac{1}{N-1}}$$

The parameter N determines the shape of the nonlinear function, and M, b are scaling constants. This function was discussed by Kipinick (26), who used the sharp breaking feature (with large N) in a penalty function application. In the present application it provides a convenient nonlinear function which can be altered over a wide range of shapes.

The cost index is taken to be

$$J[u] = \int_0^T (q_1x_1^2 + q_2x_2^2 + u^2)dt \tag{B-2}$$

where T is left unspecified.

The Hamiltonian function is

$$H = -(q_1x_1^2 + q_2x_2^2 + u^2) + x_3x_2 + x_4(-\omega x_2 + KF(N,u)) \tag{B-3}$$

where x_3, x_4 are the co-state variables. From the Maximum Principle the optimum control is found to satisfy

$$u^* = \operatorname{sgn}(x_4)\left[\frac{MbK}{N(N-1)}\left(\frac{2b}{N}\right)^{-\frac{(N-2)}{(N-1)}} \left|x_4\right|\right]^{\frac{(N-1)}{2(N-3)}} \tag{B-4}$$

The output from the nonlinear element then becomes

$$F(N,x_4) = \operatorname{sgn}(x_4)M\left[\frac{2b^2KM}{N^2(N-1)}\left|x_4\right|\right]^{\frac{1}{(2N-3)}} \tag{B-5}$$

The necessary conditions are therefore found to be

$$\dot{x}_1 = x_2 \tag{B-6}$$

$$\dot{x}_2 = -\omega x_2 + KF(N,x_4)$$

$$\dot{x}_3 = 2q_1x_1$$

$$\dot{x}_4 = 2q_2x_2 - x_3 + \omega x_4$$

The five free time, two point boundary conditions for (B-6) are

$$x_1(0) = x_{10} \qquad x_1(T) = 0 \qquad H = 0 \tag{B-7}$$

$$x_2(0) = x_{20} \qquad x_2(T) = 0$$

where H is given by B-3.

The adjoint set to B-7 is

$$\dot{\lambda}_1 = -2q_1\lambda_3 \tag{B-8}$$

$$\dot{\lambda}_2 = -\lambda_1 + \omega\lambda_2 - 2q_2\lambda_4$$

$$\dot{\lambda}_3 = \lambda_4$$

$$\dot{\lambda}_4 = -K\frac{\partial F(N,x_4)}{\partial x_4}\lambda_2 - \omega\lambda_4$$

However, the severity of the nonlinear function (for large N) made solution by the adjoint method very inefficient. Conversely, the flooding technique was very effective. Runge-Kutta integration was employed. The variable step size error control scheme played a vital role.

The boundary conditions (B-7) establish the starting conditions for the back integrations. Thus at the start

$$H\Big|_{x(T)=0} = -u^2 + Kx_4F(N,u) = 0$$

Since the expression is independent of x_3, the choice of this variable is arbitrary. For the simpler linear case N=2 the above reduces to the requirement

$$-(1 - \frac{Mb}{2})\, x_4^2(T) = 0$$

This cannot be satisfied in general unless $x_4(T)=0$. Similarly, when $N > 2$ an involved expression is found which cannot be satisfied in general for non-zero $x_4(T)$. Thus, the following starting values are used for the flooding trajectories

$$x_1(T) = 0 \qquad x_3(T) \neq 0 \tag{B-9}$$

$$x_2(T) = 0$$

$$x_4(T) = 0$$

The plant constants are taken as

$$K = 1 \tag{B-10}$$

$$\omega = 0.3$$

The value of ω, in rad./sec., is typical of the inertia and damping loads found on large inertial platforms. The scaling constants M, b are chosen for each N to normalize the nonlinear function such that $F(N,u)=1$ at $u=1$.

APPENDIX C

CASE THREE FORMULATION

The differential constraint equations for the Case Three plant are

$$\dot{x}_1 = x_2 \qquad \text{(C-1)}$$

$$\dot{x}_2 = K_2 T[n_2(\dot{\Theta}_r + x_2),\ K_p(\Theta_{p_r} + x_3)]$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\omega x_4 + K_1 u$$

$$K_1 = \frac{n_1}{J_1} K_m \qquad \omega = \frac{n_1^2 D}{J_1}$$

$$K_2 = \frac{n_2}{J_2}$$

The motor drive torque function $T[x_2, x_3]$ is a nonlinear function of speed and input voltage. The appropriate equations are given farther on in this appendix. A linearized version of the torque equations was also included as an option in the computer program.

The elements of the state vector x are defined as

$$x_1 = \Theta - \Theta_r \qquad \dot{\Theta}_r = \text{constant speed reference input} \tag{C-2}$$

$$x_2 = \dot{\Theta} - \dot{\Theta}_r$$

$$\Theta_{p_r} = \Theta_p \Big|_{\dot{\Theta}=\dot{\Theta}_r} = \text{constant}$$

$$x_3 = \Theta_p - \Theta_{p_r}$$

$$x_4 = \dot{\Theta}_p$$

where Θ is the output shaft position and Θ_p is pilot shaft (pot) position.

The cost index is

$$J[u] = \int_0^T (x^T Q x + u^2) dt \tag{C-3}$$

where Q is a symmetric 4x4 matrix and T may be fixed or free.

The Hamiltonian function is

$$\begin{aligned} H = -[& q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + q_{44}x_4^2 + \\ & 2(q_{23}x_2x_3 + q_{24}x_2x_4 + q_{34}x_3x_4) + u^2] + \\ & x_5x_2 + K_2x_6T[n_2(\dot{\Theta}_r + x_2), K_p(\Theta_{p_r} + x_3)] + \\ & x_7x_4 + x_8(-\omega x_4 + K_1u) = 0 \end{aligned} \tag{C-4}$$

where x_5, x_6, x_7, x_8 are the co-state variables. From the Maximum Principle the open-loop control is

$$u^* = \frac{K_1^2}{2} x_8 \tag{C-5}$$

From (C-4),(C-5) the following necessary conditions can be derived

$$\dot{x}_1 = x_2 \tag{C-6}$$

$$\dot{x}_2 = K_2 T[n_2(\dot{\Theta}_r + x_2),\ K_p(\Theta_{p_r} + x_3)]$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\omega x_4 + \frac{K_1^2}{2} x_8$$

$$\dot{x}_5 = 2q_{11}x_1$$

$$\dot{x}_6 = 2(q_{22}x_2 + q_{23}x_3 + q_{24}x_4) - x_5 - K_2 x_6 \frac{\partial T}{\partial x_2}$$

$$\dot{x}_7 = 2(q_{33}x_3 + q_{23}x_2 + q_{34}x_4) - K_2 x_6 \frac{\partial T}{\partial x_3}$$

$$\dot{x}_8 = 2(q_{44}x_4 + q_{24}x_2 + q_{34}x_3) - x_7 + \omega x_8$$

The appropriate two point boundary conditions are

$$g(0) = 0 \tag{C-7}$$

where

$$g_1 = x_1(0) - x_{10}$$

$$g_2 = x_2(0) - x_{20}$$

$$g_3 = x_3(0) - x_{30}$$

$$g_4 = x_4(0) - x_{40}$$

and

$$h(T) = 0$$

where

$$h_1 = x_1(T)$$

$$h_2 = x_2(T)$$

$$h_3 = x_3(T)$$

$$h_4 = x_4(T)$$

$$h_5 = -[q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + q_{44}x_4^2 + 2(q_{23}x_2x_3+q_{24}x_2x_4+q_{34}x_3x_4) - \frac{K_1^2}{4}x_8^2] + x_2x_5 + K_2x_6T[n_2(\dot{\Theta}_r + x_2), K_p(\Theta_{p_r} + x_3)] + x_4x_7 - \omega x_4x_8$$

In the fixed time (T) problem the h_5 element is deleted.

The adjoint equations to (C-6) are

$$\dot{\lambda}_1 = -2q_{11}\lambda_5 \tag{C-8}$$

$$\dot{\lambda}_2 = -\lambda_1 - K_2 \frac{\partial T}{\partial x_2} \lambda_2 - (2q_{22} - K_2 x_6 \frac{\partial^2 T}{\partial x_2^2})\lambda_6$$

$$- (2q_{23} - K_2 x_6 \frac{\partial^2 T}{\partial x_2 \partial x_3})\lambda_7 - 2q_{24}\lambda_8$$

$$\dot{\lambda}_3 = -K_2 \frac{\partial T}{\partial x_3} - (2q_{23} - K_2 x_6 \frac{\partial^2 T}{\partial x_3 \partial x_2})\lambda_6$$

$$- (2q_{33} - K_2 x_6 \frac{\partial^2 T}{\partial x_3^2})\lambda_7 - 2q_{34}\lambda_8$$

$$\dot{\lambda}_4 = -\lambda_3 + \omega\lambda_4 - 2q_{24}\lambda_6 - 2q_{34}\lambda_7 - 2q_{44}\lambda_8$$

$$\dot{\lambda}_5 = \lambda_6$$

$$\dot{\lambda}_6 = K_2 \frac{\partial T}{\partial x_2} \lambda_6 + K_2 \frac{\partial T}{\partial x_3} \lambda_7$$

$$\dot{\lambda}_7 = \lambda_8$$

$$\dot{\lambda}_8 = \frac{-K_1^2}{2} \lambda_4 - \omega\lambda_8$$

When the adjoint method is used to solve the two point boundary value problem the time derivative vector $\dot{h}$ is required.

$$\dot{h}_1 = x_2 \tag{C-9}$$

$$\dot{h}_2 = K_2 T[n_2(\dot{\Theta}_r + x_2), K_p(\Theta_{p_r} + x_3)]$$

$$\dot{h}_3 = x_4$$

$$\dot{h}_4 = -\omega x_4 + \frac{K_1^2}{2} x_8$$

$$\dot{h}_5 = 0$$

Furthermore, the following quantities must be supplied to the general integration program, in the free time problem, to augment the adjoint initial conditions supplied automatically by the program.

$$\frac{\partial h_5}{\partial x_1} = -2q_{11}x_1 \tag{C-10}$$

$$\frac{\partial h_5}{\partial x_2} = -2(q_{22}x_2 + q_{23}x_3 + q_{24}x_4) + x_5 + K_2 x_6 \frac{\partial T}{\partial x_2}$$

$$\frac{\partial h_5}{\partial x_3} = -2(q_{33}x_3 + q_{23}x_2 + q_{34}x_4) + K_2 x_6 \frac{\partial T}{\partial x_3}$$

$$\frac{\partial h_5}{\partial x_4} = -2(q_{44}x_4 + q_{24}x_2 + q_{34}x_3) + x_7 - \omega x_8$$

$$\frac{\partial h_5}{\partial x_5} = x_2$$

$$\frac{\partial h_5}{\partial x_6} = K_2 T[n_2(\dot{\Theta}_r + x_2), K_p(\Theta_{p_r} + x_3)]$$

$$\frac{\partial h_5}{\partial x_7} = x_4$$

$$\frac{\partial h_5}{\partial x_8} = \frac{K_1^2}{2} x_8 - \omega x_4$$

If the flooding alternative is used in the solution of the two point boundary value problem, a choice for the co-states $x_5(T)$, $x_6(T)$, $x_7(T)$, $x_8(T)$ must be made to satisfy the boundary conditions (C-7). In the fixed time problem the choice is unconstrained. In the free time case the condition $h_5(T)=0$ gives the requirement

$$x_6(T) = \frac{-K_1^2}{4K_2 T[n_2\dot{\Theta}_r, K_p\Theta_{p_r}]} x_8^2(T) \qquad (\dot{\Theta}_r \neq 0) \tag{C-11}$$

$$= x_8(T) = 0 \qquad (\dot{\Theta}_r = 0)$$

$$T = \text{free}$$

with all other co-state selections unconstrained.

Motor Torque Equations

Over a wide range of speed and input volts, the torque equations of the two phase servomotor are nonlinear. Relationships given by Koopman[1] lead to the expression

$$T(s,k) = T(s,1)\left(\frac{V_{m_1}}{V_m}\right)^2 - T(-s,1)\left(\frac{V_{m_2}}{V_m}\right)^2$$

---

[1] Ref. 31, eqns. 20, 21, 22.

$$s = \frac{\dot{\Theta}}{\dot{\Theta}_m} \quad : \text{synchronous speed ratio}$$

$$k = \frac{E_c}{V_m} \quad : \text{control voltage ratio}$$

$$V_{m_1} = \frac{V_m}{2}(1 + k)$$

$$V_{m_2} = \frac{V_m}{2}(1 - k)$$

This equation corresponds to the case of a near zero source impedance to the control winding, a case often found in practice. Per standard methods in polyphase induction motor analysis,[1] the full rated voltage torque terms can be put in the form

$$T(s,k) = \frac{c_1(1 - s)}{[1 + c_2(1 - s)^2]}$$

where c_1, c_2 are constants based on the rated voltage, rotor resistance and inductance, and the carrier frequency. Combining these two relationships gives the required torque expression as

$$T(s,k) = \frac{c_1}{4}\left\{\frac{(1 - s)(1 + k)^2}{[1 + c_2(1 - s)^2]} - \frac{(1 + s)(1 - k)^2}{[1 + c_2(1 + s)^2]}\right\} \qquad \text{(C-12)}$$

Usually, for a selected motor the manufacturer makes available a torque-speed curve at full rated voltage, at least over the positive torque-speed-

[1]See e.g. Ref. 32, eqn. 28, pp 12.

volt regime. This data can be used to determine appropriate numerical values for the constants c_1, c_2. Suppose $T(0,1) = T_{01}$, $T(\sigma,1) = T_{11}$, two selected rated volt data points at stall and speed ratio $S=\sigma>0$. Then in terms of these data points the following expressions for c_1, c_2 can be derived from (C-12).

$$c_1 = \frac{2\sigma(2-\sigma)T_{01}T_{11}}{(1-\sigma)[T_{01}-(1-\sigma)T_{11}]} \tag{C-13}$$

$$c_2 = \frac{T_{11}-(1-\sigma)T_{01}}{(1-\sigma)[T_{01}-(1-\sigma)T_{11}]}$$

The torque data is taken for a single motor. The constant c_1 has been adjusted to reflect the tandem drive motors of this problem. Since (C-12) is a theoretical equation, the constants (C-13) will give exact agreement of the torque equation and the experimental data only at the two selected data points. However, for the motor selected for this problem, good agreement was found possible with (C-12) over a wide range of operating conditions, as will be shown farther on.

A number of torque partial derivatives have appeared in some of the preceding analysis. These terms can now be derived from (C-12). In the form required by the analysis these are

$$\frac{\partial T}{\partial x_2} = \frac{-c_1 n_2}{4\dot{\Theta}_m}\left\{\frac{[1-c_2(1-s)^2](1+k)^2}{D_1^2} + \frac{[1-c_2(1+s)^2](1-k)^2}{D_2^2}\right\} \tag{C-14}$$

$$\frac{\partial T}{\partial x_3} = \frac{c_1 K_p}{2V_m}\left\{\frac{(1-s)(1+k)}{D_1} + \frac{(1+s)(1-k)}{D_2}\right\}$$

$$\frac{\partial^2 T}{\partial x_2^2} = \frac{-c_1c_2n_2^2}{2\dot{\Theta}_m^2}\left\{\frac{(1-s)(1+k)^2[3-c_2(1-s)^2]}{D_1^3} - \frac{(1+s)(1-k)^2[3-c_2(1+s)^2]}{D_2^3}\right\}$$

$$\frac{\partial^2 T}{\partial x_3 \partial x_2} = \frac{\partial^2 T}{\partial x_2 \partial x_3} = \frac{-c_1 n_2 K_p}{2V_m\dot{\Theta}_m}\left\{\frac{(1+k)[1-c_2(1-s)^2]}{D_1^2} - \frac{(1-k)[1-c_2(1+s)^2]}{D_2^2}\right\}$$

$$\frac{\partial^2 T}{\partial x_3^2} = \frac{c_1K_p^2}{2V_m^2}\left\{\frac{(1-s)}{D_1} - \frac{(1+s)}{D_2}\right\}$$

$$D_1 = 1 + c_2(1-s)^2$$

$$D_2 = 1 + c_2(1+s)^2$$

The torque relations can be linearized for use when this option is applicable. For the linearized case the following expressions are used:

$$T(s,k) = 2[-(D\dot{\Theta}_m)s + (K_mV_m)k] \qquad \text{(C-15)}$$

$$\frac{\partial T}{\partial x_2} = -2D\,n_2$$

$$\frac{\partial T}{\partial x_3} = 2K_pK_m$$

All second partial derivatives are zero by definition.

The quantity Θ_{p_r} is the value of Θ_p required to maintain the command speed $\dot{\Theta}_r$ in steady state, and will depend on the magnitude of any load torque on the output shaft. In the physical system the value of Θ_{p_r} is arrived at automatically by the feedback loop. For computational purposes it must be derived from an inverse relation of the output motor torque equation, either analytically or graphically from the torque-speed-volt curves for the motor. In the case of the linearized system the relation is

$$\Theta_{p_r} = \frac{n_2 D\dot{\Theta}_r - T_{load}}{K_p K_m} \qquad \text{(C-16)}$$

Phase Variables

An expression relating the phase variables to the selected state variables is found useful. Let y be the phase variable vector defined by

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \Theta \\ \dot{\Theta} \\ \ddot{\Theta} \\ \dddot{\Theta} \end{bmatrix} \qquad \text{(C-17)}$$

To express phase (y) as a function of the state (x), (C-2) and (C-17) are used to derive

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \Theta_r + x_1 \\ \dot{\Theta}_r + x_2 \\ K_2 T[n_2(\dot{\Theta}_r + x_2),\ K_p(\Theta_{p_r} + x_3)] \\ K_2 \left\{ \frac{\partial T}{\partial x_2} K_2 T[n_2(\dot{\Theta}_r + x_2),\ K_p(\Theta_{p_r} + x_3)] + \frac{\partial T}{\partial x_3} x_4 \right\} \end{bmatrix} \tag{C-18}$$

Thus in general a nonlinear algebraic transformation is required. A special case of interest results when the system is linear. Then

$$K_2 T(x_2, x_3) = -\omega_2(\dot{\Theta}_r + x_2) + K_a(\Theta_{p_r} + x_3)$$

$$\omega_2 = 2Dn_2K_2$$

$$K_a = 2K_mK_2K_p$$

which gives the linear form

$$y = Mx + b \tag{C-19}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\omega_2 & K_a & 0 \\ 0 & \omega_2^2 & -\omega_2 K_a & K_a^2 \end{bmatrix}, \qquad b = \begin{bmatrix} \Theta_r \\ \dot{\Theta}_r \\ -\omega_2\dot{\Theta}_r \\ K_a\Theta_{p_r} \end{bmatrix}$$

If $\Theta_r = \dot{\Theta}_r = 0$ this reduces to the very convenient result

$$y = Mx \tag{C-20}$$

Plant Constants

The following plant constants must be specified: J_1, J_2, n_1, n_2, K_p, D, K_m, $\dot{\theta}_m$, V_m, c_1, c_2 . In a real application a typical set of values might have been established as follows.

Using units of ounce-inch, degrees, seconds, suppose the output load inertia (J_{L_2}) is comprised of a rather heavy mass such that

$$J_{L_2} = 20{,}000 \text{ oz.-in.}^2 = 0.905 \text{ oz.-in.-sec.}^2/\text{deg.}$$

A selection of Diehl Type SSFPE 25-11 five watt servomqtors is made. Then from catalogue data[1] each motor has the inertia

$$J_m = 0.077 \text{ oz.-in.}^2 = 3.48 \times 10^{-6} \text{ oz.-in.-sec.}^2/\text{motor deg.}$$

For a maximum output acceleration capability, use the rule-of-thumb

$$n_2 \approx \sqrt{J_L/J_m} \cong 360$$

Mechanical design economy, simplicity, and a high tracking velocity requirement overrule, however, to give the compromise ratio

$$n_2 = 150$$

Assuming the train inertia is lumped in the J_{L_2} figure, the total equivalent inertia at the load shaft is

[1]Ref. 33, Table 9-1, p. 140.

$$J_2 = J_{L_2} = n_2^2(2J_m) = 1.06 \text{ oz.-in.-sec.}^2/\text{deg.}$$

A ten-turn ITC M-10-T-19 pot is selected as the pilot motor transducer. Reference 34 lists

$$J_p = 15 \text{ gm.-cm.}^2 = 3.72 \times 10^{-6} \text{ oz.-in.-sec.}^2/\text{deg.}$$

Suppose the pilot drive and special limit stops add an additional inertia of

$$J_d = 0.05 \text{ oz.-in.}^2 = 2.26 \times 10^{-6} \text{ oz.-in.-sec.}^2/\text{deg.}$$

On the basis of matching pilot motor and load inertias,

$$n_1 \simeq \sqrt{\frac{J_p + J_d}{J_m}} = 1.31$$

However, since pilot acceleration is not expected to be critical, the following higher ratio is settled upon to increase sensitivity and to reduce jitter and threshold errors.

$$n_1 = 50$$

Then the total pilot load inertia <u>at the pot shaft</u> is

$$J_1 = J_p + J_d + n_1^2 J_m = 8.71 \times 10^{-3} \text{ oz.-in.-sec.}^2/\text{deg.}$$

To provide high resolution from the pot-servoamp combination set

$$K_p = 0.1$$

The five watt Diehl motors have the further characteristics

$$V_m = 115 \text{ volts}$$

$$\dot{\theta}_m = 21{,}000 \text{ deg./sec.}$$

A set of torque-speed-volt curves are given in Ref. 33[1]. Two data points were taken from these curves such that

$$T_{01} = 5.60 \text{ oz.-in.}$$

$$T_{11} = 3.88 \text{ oz.-in.} \quad (\sigma = 1/2)$$

Then from (C-13) the nonlinear torque equation constants c_1, c_2 are computed as

$$c_1 = 17.80$$

$$c_2 = 0.590$$

Figure 37 shows a plot of the nonlinear torque equation (C-12) with these constants. Some points from the experimental data are included to show the excellent agreement with the theoretical curves. Several other sets of constants were computed from alternate data points but the results were not as good as those of the elected pair.

In a linearized analysis it is common practice to choose the motor constants from the low speed-volt region of operation. On this basis the following linearized motor constants were selected from the experimental data.

---

[1]Figure 9-14, p. 138.

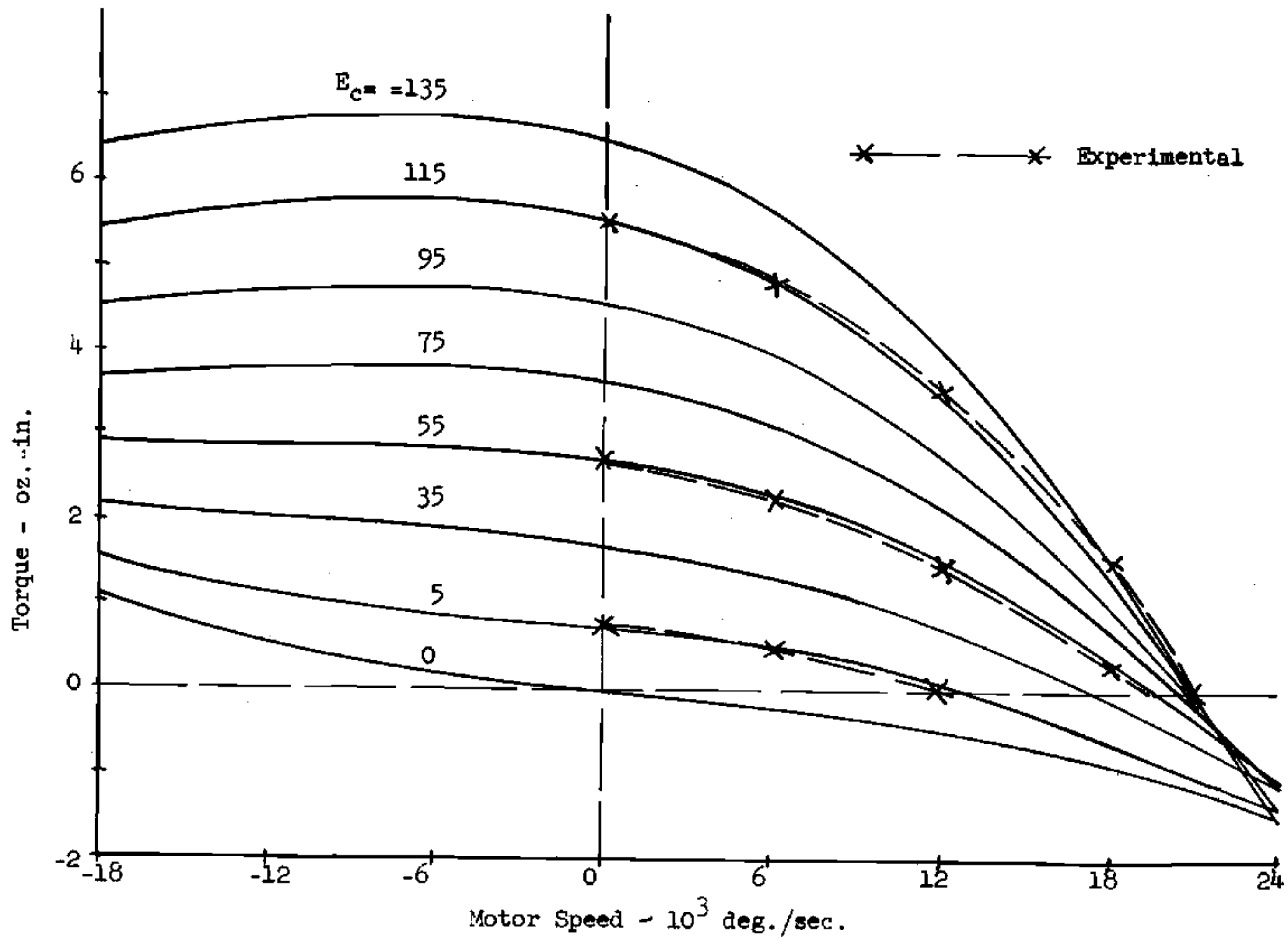


Fig. 37 Motor Torque-Speed-Volt Data

$$D = 0.782 \times 10^{-4} \text{ oz.-in.-sec./motor deg.}$$

$$K_m = 0.0475 \text{ oz.-in./volt}$$

An alternate condition is also considered where the damping in the pilot motor is negligible $(D_1 = 0)$. This corresponds to a torque motor type of drive.

In a linearized analysis, the open loop transfer function will be of the form

$$G(s) = \frac{\Theta}{u} = \frac{K_o}{s^2(s + \omega_1)(s + \omega_2)} \tag{C-21}$$

For the selected constants the break frequencies are

$$\omega_1 = \frac{n_1^2 D_1}{J_1} = 22.4 \text{ rad./sec.} \quad (D_1 = 0.782 \times 10^{-4})$$

$$= 0 \qquad (D_1 = 0)$$

$$\omega_2 = \frac{2D_2 n_2^2}{J_2} = 3.32 \text{ rad./sec.}$$

The plant is now completely specified. The required constants are summarized below.

$$J_1 = 8.71 \times 10^{-3} \text{ oz.-in.-sec.}^2\text{/deg.} \tag{C-22}$$

$$J_2 = 1.06 \text{ oz.-in.-sec.}^2\text{/deg.}$$

$$n_1 = 50$$

$n_2 = 150$

$K_p = 0.1$ volt/deg.

$D = 0,\ 0.782 \times 10^{-4}$ oz.-in.-sec./motor-deg.

$K_m = 0.0475$ oz.-in./volt

$\dot{\Theta}_m = 21{,}000$ deg./sec.

$V_m = 115$ volts

$c_1 = 17.80$

$c_2 = 0.590$

From these values

$K_1 = 272$ motor deg./volt-sec.^{2}

$K_2 = 141.5$ motor deg./oz.-in.-sec.^{2}

$$\omega = \omega_1 = 22.4 \text{ sec}^{-1} \quad (D_1 = .782 \times 10^{-4})$$
$$= 0 \qquad\qquad (D_1 = 0)$$

It is emphasized that all plant design constants have been derived in a way which might typically occur in practice, and therefore in this sense, they are not arbitrary.

APPENDIX D

LISTING OF GENERAL NUMERICAL INTEGRATION PROGRAM

A brief description of the integration program was given in Chapter II. Because of the importance of the program in generating all open and closed-loop trajectories, a listing of the basic program and a sample set of subroutines is recorded in this section. All programing was done in the more settled FORTRAN IV language. Actual runs were made from Burroughs translations into Extended ALGOL for their B-5000/5500 machines at Georgia Institute of Technology. Because of the rather fluid state of the Burroughs hardware and translators over the past several years, the equivalent ALGOL listings used to generate the results have varied accordingly.

```
C   GENERAL NUMERICAL INTEGRATION PROGRAM FOR INITIAL AND TWO POINT
C   BOUNDARY VALUE PROBLEMS WITH TYPICAL SUBROUTINE SET.
C
C   MAIN PROGRAM
      DIMENSION X(10,2,200),DX(10),DXD(10),CC(10,4),CI(10),XI(10),XF(10)
      DIMENSION XD(10,2,200),XX(10),XXS(10,10),ADZ(5,10),HF(10),HFD(10)
      DIMENSION AA(5,10),MBTI(10),MBTF(10),CH(10),PHF(10),XIS(10)
      DIMENSION U1(200),T1(200),T2(200),B(20),HFS(10),HFDS(10)
      WRITE              (3,1)
      MRUN=0
      NDIMB=0
C
C   DECLARE CARD INSERT OPTIONS JTYPE, IMODE, AND MOUT HERE
      JTYPE=1
      JTYPE=2
      IMODE=1
      IMODE=2
      MOUT=1
      MOUT=2
      GO TO(92,95),JTYPE
   92 GO TO(93,94),IMODE
   93 WRITE              (3,2)
      GO TO 98
   94 WRITE              (3,15)
      GO TO 98
   95 GO TO(96,97),IMODE
   96 WRITE              (3,16)
      GO TO 89
   97 WRITE              (3,17)
   89 GO TO(90,91),MOUT
   90 WRITE              (3,18)
      GO TO 98
   91 WRITE              (3,19)
   98 NP=0
      WRITE              (3,3)
C
C   INSERT STANDARD DATA SUBROUTINE CALLING HERE
      CALL STDATA(NDIM,NDIMB,ERLMT,NPUT,HEXT,HINT,GA,GB,GR1,GR2,
     1  W1,W2,C1,C2,C3,C4,XF,JTYPE,B,NLOP,GP,OMG,DD)
      IF(NDIMB)70,86,100
   70 NBL=1
      GO TO 183
   86 NDIMB=NDIM
  100 NDIMS=NDIM
C
C   STEP SIZE INITIALIZATION. (CMAX AND DCI)
      CMAX=1.
      DCI=0.2
      ITMAX=10
      MRUN=MRUN+1



      WRITE              (3,4)MRUN
      DO 99 I=1,NDIM
   99 CI(I)=CMAX
C
C   INSERT INPUT DATA READ/WRITE SUBROUTINE CALLING HERE
      CALL INPUT(XI,HEXT,HINT,NFNL,NDAT,JTYPE,B,NLOP,THRD,THPR,W1,W2,
     1  GR2,GP)
      KFAULT=0
      ITER=0
      INCR=0
      RMBL=1.E38
      ISWP=0
      NHH=0
      NRED=10
      EMA=5
  105 J=1
      M=1
      T=0.
      N=1
      RMSER=0.
      KINK=1
      LINK=1
C
C   INSERT BOUNDARY VALUE SUBROUTINE CALLING FOR LINK=1 HERE (1 OF 2)
      CALL BOUNDR(LINK,NP,NPT,XI,XF,N,X,HF,HFD,MBTI,MBTF,PHF,C1,C2,C3,
     1  C4,GA,GB,OMG,GR2,GP,W1,W2,THRD,THPR,NLOP,DD)
C   NUMERICAL INTEGRATION SECTION
  300 GO TO (301,320),IMODE
C   MODIFIED EULER INTEGRATION
  301 DO 302 I=1,NDIM
  302 DX(I)=X(I,J,N)
C
C   INSERT DIFFERENTIAL EQNS. SUBR. CALLING HERE (1 OF 3)
      CALL DFEQNS(DX,DXD,J,X,N,GA,GB,GR1,GR2,W1,W2,C1,C2,C3,C4,
     1  GP,HEXT,HINT,B,THRD,THPR,OMG,NLOP,JTYPE,DD)
      DO 309 I=1,NDIM
  309 XD(I,J,N)=DXD(I)
  310 DO 311 I=1,NDIM
  311 DX(I)=X(I,J,N)+HEXT*DXD(I)
C
C   INSERT DIFFERENTIAL EQNS. SUBR. CALLING HERE (2 OF 3)
      CALL DFEQNS(DX,DXD,J,X,N,GA,GB,GR1,GR2,W1,W2,C1,C2,C3,C4,
     1  GP,HEXT,HINT,B,THRD,THPR,OMG,NLOP,JTYPE,DD)
      DO 315 I=1,NDIM
      X(I,J,N+M)=X(I,J,N)+HEXT*(DXD(I)+XD(I,J,N))/2.
  315 XD(I,J,N+M)=DXD(I)
      GO TO 350
C   RUNGE-KUTTA INTEGRATION
  320 DO 329 JINC=1,4
      GO TO(321,322,324,323),JINC
```

```
  321 CRK=0.
      INC=1
      GO TO 324
  322 CRK=0.5
      GO TO 324
  323 CRK=1.0
  324 DO 325 I=1,NDIM
  325 DX(I)=X(I,J,N)+CRK*CC(I, INC)
C
C  INSERT DIFFERENTIAL EQNS. SUBR. CALLING HERE (3 OF 3)
      CALL DFEQNS(DX,DXD,J,X,N,GA,GB,GR1,GR2,W1,W2,C1,C2,C3,C4,
     1  GP,HEXT,HINT,B,THRD,THPR,OMG,NLOP,JTYPE,DD)
      INC=JINC
      DO 329 I=1,NDIM
  329 CC(I,JINC)=HEXT*DXD(I)
C  RUNGE-KUTTA ERROR CONTROL.  TO DELETE GO TO 505.
      DO 510 I=1,NDIM
      DM=ABS(CC(I,2)-CC(I,1))
      DN=ABS(CC(I,3)-CC(I(2))
  501 IF(DN-EMA*DM)510,510,502
  510 CONTINUE
      IF(NHH)507,505,507
  507 NHH=NHH+1
      IF(NHH-NRED)506,506,511
  511 NHH=0
      HEXT=HH
      GO TO 506
  500 IF(DN-DM)501,501,512
  512 WRITE (3,24)DM,N,NHH
      GO TO 501
  502 IF(NHH)513,504,513
  513 IF(NHH-1)514,514,507
  514 WRITE (3,26)DN,DM,N,NHH
      GO TO 507
  504 WRITE (3,IK)DN,DM,N,NRED,NRED
      HH=HEXT
      RED=NRED
      HEXT=HEXT/RED
      NHH=1
      DO 515 I=1,NDIM
  515 X(I,J,N+M)=X(I,J,N)
      GO TO 320
  506 DO 516 I=1,NDIM
  516 X(I,J,N+M)=X(I,J,N+M)+(CC(I,1)+2*(CC(I,2)+CC(I,3))+CC(I,4))/6
      IF (NHH)320,350,320
  505 CONTINUE
      DO 330 I=1,NDIM
  330 X(I,J,N+M)=X(I,J,N)+(CC(I,1)+2.*(CC(I,2)+CC(I,3))+CC(I,4))/6.
C  INTEGRATION MONITOR SECTION
  350 GO TO(360,361),J

  360 GO TO(351,358),KINK
  351 IF(NFNL-(N+1))71,356,352
   71 NBL=2
      GO TO 183
  356 IF(HINT)357,359,357
  357 HEX=HEXT
      RN=N
      T=RN*HEXT
      N=NFNL
      HEXT=HINT
      KINK=2
      GO TO(301,320),IMODE
  352 RN=N
      T=RN*HEXT
      N=N+1
      GO TO(301,320),IMODE
  358 RN=NFNL-1
      T=RN*HEX+HINT
      N=NFNL+1
      HEXT=HEX
      GO TO 353
  359 RN=N
      T=RN*HEXT
      N=NFNL
  353 GO TO(190,120),JTYPE
  361 GO TO(354,362),KINK
  362 GO TO(363,354),KINC
  363 HEXT=-HEX
      KINC=2
  354 IF(N-2)72,151,355
   72 NBL=3
      GO TO 183
  355 N=N-1
      RN=N-1
      T=RN*HEXT
      GO TO(301,320),IMODE
C  CONVERGENCE CONTROL SECTION
  120 LINK=2
C
C  INSERT BOUNDARY VALUE SUBROUTINE CALLING FOR LINK=2 HERE (2 OF 2)
      CALL BOUNDR(LINK,NP,NPT,XI,XF,N,X,HF,HFD,MBTI,MBTF,PHF,C1,C2,C3,
     1  C4,GA,GB,OMG,GR2,GP,W1,W2,THRD,THPR,NLOP,DD)
      DO 121 I=1,NPT
  121 RMSER=RMSER+HF(I)*HF(I)
      RMSER=SQRT (RMSER)
      GO TO (131,190),MOUT
  137 IF(RMSER-ERLMT)138,138,122
  138 WRITE              (3,20)
      GO TO 100
  131 IF(RMSER-ERLMT)190,190,122
```

```
  122 IF(RMSL-5.E37)124,124,123
  123 RMSL=RMSER
      ITER=1
      ISWP=1
      GO TO 150
  124 ISWP=ISWP+1
      IF(RMSER-RMSL)125,125,129
  125 RMSL=RMSER
      IF(ITER-ITMAX)127,182,182
  127 IF(ISWP-30)134,182,182
  134 ITER=ITER+1
      IF(INCR)73,150,132
   73 NBL=4
      GO TO 183
  129 INCR=INCR+1
      RINCR=INCR
      DO 130 I=1,NP
      XXS(INCR,I)=XX(I)
      NN=MBTI(I)
      XI(NN)=XIS(NN)
      HF(I)=HFS(I)
      HFD(I)=HFDS(I)
  130 CI(I)=CMAX-RINCR*DCI
      XXS(INCR,NPT)=XX(NPT)
      HF(NPT)=HFS(NPT)
      HFD(NPT)=HFDS(NPT)
      NFNL=NFNLS
      HINT=HINTS
      CI(NPT)=CMAX-RINCR*DCI
      IF(INCR-4)156,156,180
  132 INCR=INCR-1
      RINCR=INCR
      DO 133 I=1,NPT
  133 CI(I)=CMAX-RINCR*DCI
      GO TO 150
  135 DO 136 I=1,NP
      NN=MBTI(I)
      XIS(NN)=XI(NN)
      HFS(I)=HF(I)
      HFDS(I)=HFD(I)
  136 XI(NN)=XI(NN)+XX(I)
      NFNLS=NFNL
      HINTS=HINT
      IF(RMSER-RMSL)141,141,140
  141 HEXT=-HEXT
  140 IF(NPT-NP)74,148,139
   74 NBL=5
      GO TO 183
  139 IF(HINT+XX(NPT))145,144,144
  144 NXX=(HINT+XX(NPT))/HEXT

      NFNL=NFNL+NXX
      RNXX=NXX
      HINT=HINT+XX(NPT)-RNXX*HEXT
      GO TO 148
  145 IF(HEXT+HINT+XX(NPT))147,146,146
  146 NFNL=NFNL-1
      HINT=HEXT+HINT+XX(NPT)
      GO TO 148
  147 NXX=(HINT+XX(NPT))/HEXT
      NFNL=NFNL+NXX-1
      RNXX=NXX
      HINT=HEXT+HINT+XX(NPT)-RNXX*HEXT
  148 HFS(NPT)=HF(NPT)
      HFDS(NPT)=HFD(NPT)
      GO TO 105
C  MATRIX INVERSION SECTION (GAUSS JORDAN)
  400 DO 410 I=1,NPT
  410 CH(I)=-CI(I)*HF(I)
      DO 404 I=1,NPT
      IF(AA(I,I))401,781,401
  781 MI=I
      GO TO 181
  401 CH(I)=CH(I)/AA(I,I)
      MX=I+1
      IF(MX-NPT)407,407,408
  407 DO 402 K=MX,NPT
  402 AA(I,K)=AA(I,K)/AA(I,I)
  408 AA(I,I)=1.
      IF(MX-NPT)409,409,404
  409 DO 404 L=MX,NPT
      CH(L)=CH(L)-AA(L,I)*CH(I)
      DO 403 K=MX,NPT
  403 AA(L,K)=AA(L,K)-AA(L,I)*AA(I,K)
      AA(L,I)=0.
  404 CONTINUE
      DO 405 L=1,NPT
  405 XX(NPT-L+1)=CH(NPT-L+1)
      MY=NPT-1
      IF(MY)135,135,411
  411 DO 406 L=1,MY
      DO 406 LL=1,L
  406 XX(NPT-L)=XX(NPT-L)-AA(NPT-L,NPT-LL+1)*XX(NPT-LL+1)
      GO TO 135
  180 WRITE             (3,6)INCR
      KFAULT=1
      DO 185 K=1,4
      WRITE             (3,9)K
      DO 185 I=1,NPT
  185 WRITE             (3,10)I,XXS(K,I)
      K=5
```

```
      WRITE               (3,9)K
      DO 186 I=1,NPT
  186 WRITE               (3,11)I,XXS(5,I),I,HF(I),I,HFD(I)
      GO TO(190,100),MOUT
  181 WRITE               (3,7)M1,M1
      KFAULT=1
      DO 187 I=1,NPT
  187 WRITE               (3,12)I,ADZ(1,I),I,HF(I),I,HFD(I)
      NY=NPT+1
      DO 195 I=NY,NDIM
  195 WRITE               (3,21)I,ADZ(1,I)
      DO 188 K=2,NPT
      DO 188 I=1,NDIM
  188 WRITE               (3,13)K,I,ADZ(K,I)
      GO TO(190,100),MOUT
  182 WRITE               (3,8)
      KFAULT=1
      DO 189 I=1,NPT
  189 WRITE               (3,14)I,XX(I),I,HF(I),I,HFD(I)
      GO TO(190,100),MOUT
  183 IF(NFNL)184,196,196
  184 WRITE               (3,23)NFNL,HINT
      GO TO 197
  196 WRITE               (3,22)NBL
  197 GO TO(190,100),MOUT
  150 J=2
      M=-1
      HEXT=-HEXT
      IBCK=1
      GO TO(168,169),KINK
  168 N=NFNL
      GO TO 154
  169 N=NFNL+1
      GO TO 154
  151 GO TO(164,163),KINK
  163 N=NFNL+1
      RN=NFNL-1
      T=-RN*HEXT+HINT
      GO TO 166
  164 N=NFNL
      RN=NFNL-1
      T=-RN*HEXT
  166 DO 152 I=1,NDIM
  152 ADZ(IBCK,I)=X(I,2,1)
      NDIM=NDIMS
      IBCK=IBCK+1
  153 IF(IBCK-NP)154,154,158
  158 IF(NPT-IBCK)156,159,75
   75 NBL=6
      GO TO 183

  154 DO 155 I=1,NDIM
  155 X(I,2,N)=0.
      NN=MBTF(IBCK)
      X(NN,2,N)=1.
      NDIM=NDIMB
  165 GO TO(300,167),KINK
  167 HEXT=-HINT
      KINC=1
      GO TO 300
  159 DO 160 I=1,NDIM
  160 X(I,2,N)=PHF(I)
      GO TO 165
  156 DO 157 I=1,NPT
      DO 157 K=1,NP
      NN=MBTI(K)
  157 AA(I,K)=ADZ(I,NN)
      IF(NPT-NP)76,400,161
   76 NBL=7
      GO TO 183
  161 DO 162 I=1,NPT
  162 AA(I,NPT)=HFD(I)
      GO TO 400
  190 CONTINUE
C
C  INSERT AUXILIARY DATA SUBROUTINE CALLING HERE (IF RQD.)
      CALL AUXDTA(X,NFNL,HEXT,HINT,U1,T1,T2,GA,GB,GR1,GR2,W1,W2,
     1  C1,C2,C3,C4,GP,B,THRD,THPR,NLOP,JTYPE,NDAT,NPOINT,RMSER,NLAST,
     2  DD)
      WRITE               (3,5)ITER,ISWP,RMSER
C
C  INSERT OUTPUT DATA SUBROUTINE CALLING HERE
      CALL OUTPUT(X,NFNL,U1,T1,T2,MOUT,NPUT,MRUN,NPOINT,NDAT,
     1  NLAST,JTYPE,HEXT)
      GO TO(100,137),MOUT
    1 FORMAT(1H0,17X87HGENERAL NUMERICAL INTEGRATION PROGRAM FOR INITIAL
     1 AND TWO POINT BOUNDARY VALUE PROBLEMS)
    2 FORMAT(1H0,/,13X15HPROGRAM OPTIONS,//,17X59HINITIAL VALUE PROBLEM.
     1 MODIFIED EULER NUMERICAL INTEGRATION)
    3 FORMAT(1H0,12X13HSTANDARD DATA,/)
    4 FORMAT(1H0,12X22HINPUT DATA FOR RUN NO.,I3,/)
    5 FORMAT(1H0,/,12X13HOUTPUT DATA (,I2,14H ITERATIONS ON,I2,19H SWEEP
     1S. RMS ERROR=,E14.7,1H),/)
    6 FORMAT(1H0,//,3X64H***CONVERGENCE FAILURE. PROGRESS STOPPED DUE TO
     1 DIVERGENCE AFTER. I3,8H TRIALS.,1X36HADJUST CMAX AND/OR DCI CARD
     2INSERTS.,/,6X25HLAST DATA WAS AS FOLLOWS.,//)
    7 FORMAT(1H0,//3X31H***MATRIX INVERSION FAILURE. A(,I2,1H,I2,4H)=0.,
     1  /,7X25HLAST DATA WAS AS FOLLOWS.,//)
    8 FORMAT(1H0,3X107H***CONVERGENCE FAILURE. PROGRESS STOPPED DUE TO I
     1NSUFFICIENT CONVERGENCE RATE. RE-ESTIMATE STARTING VECTOR.,
     2  /,6X25HLAST DATA WAS AS FOLLOWS.,//)
```

```
  9 FORMAT(1H0,5X9HTRIAL NO.,I2,/)
 10 FORMAT(1H ,10X3HXX(,I2,2H)=,E14.7)
 11 FORMAT(1H ,10X3HXX(,I2,2H)=,E14.7,4X3HHF(,I2,2H)=,E14.7,
   1  3X4HHFD(,I2,2H)=,E14.7)
 12 FORMAT(1H , 9X7HADZ( 1,I2,2H)=,E14.7,4X3HHF(,I2,2H)=,E14.7,
   1  3X4HHFD(,I2,2H)=,E14.7)
 13 FORMAT(1H , 9X4HADZ(,I2,1H,,I2,2H)=,E14.7)
 14 FORMAT(1H ,10X3HXX(,I2,2H)=,E14.7,4X3HHF(,I2,2H)=,E14.7,
   1  3X4HHFD(,I2,2H)=,E14.7)
 15 FORMAT(1H0,/,13X15HPROGRAM OPTIONS,//,17X57HINITIAL VALUE PROBLEM.
   1 RUNGE KUTTA NUMERICAL INTEGRATION.)
 16 FORMAT(1H0,/,13X15HPROGRAM OPTIONS,//,17X69HTWO PT. BOUNDARY VALUE
   1 PROBLEM. MODIFIED EULER NUMERICAL INTEGRATION.)
 17 FORMAT(1H0,/,13X15HPROGRAM OPTIONS,//,17X66HTWO PT. BOUNDARY VALUE
   1 PROBLEM. RUNGE KUTTA NUMERICAL INTEGRATION.)
 18 FORMAT(1H ,/,17X18HSTANDARD PRINTOUT.)
 19 FORMAT(1H ,/,17X29HINTERMEDIATE PRINTOUT OPTION.)
 20 FORMAT(1H0,//,15X24H***SOLUTION COMPLETED***)
 21 FORMAT(1H , 9X7HADZ( 1,,I2,2H)=,E14.7)
 22 FORMAT(1H0,//,3X20H***BLUNDER ERROR NO.,I2,15H. CHECK PROGRAM)
 23 FORMAT(1H0,5X49HNEGATIVE TIME ERROR. RE-ESTIMATE STARTING VECTOR.,
   1  5X5HNFNL=,I5,3X5HHINT=,E14.7)
 24 FORMAT(1H0,3X33H***R-K ERROR CONTROL FAILURE (DM=,E14.7,7H) AT N=,
   1  I5,5H NHH=,I3,30H. CONTINUE WITHOUT ADJUSTMENT.)
 25 FORMAT(1H0,3X33H***R-K ERROR CONTROL FAILURE (DN=,E14.7,4H DM=,
   1  E14.7,7H) AT N=,I5,16H. REDUCE HEXT 1/,I3,9H FOR NEXT,I4,
   2  7H STEPS.)
 26 FORMAT(1H0,3X,53H***R-K ERROR CONTROL FAILURE ON REDUCED STEP SIZE
   1(DN=,E14.7,4H DD=,E14.7,3H N=,I5,5H NHH=,I3,/,7X43HCONTINUE UNCORR
   2ECTED WITH REDUCED STEP SIZE)
      END
C
C  TYPICAL SUBROUTINE SET (CASE THREE)
      SUBROUTINE STDATA(NDIM,NDIMB,ERLMT,NPUT,HEXT,HINT,GA,GB,GR1,GR2,
   1  W1,W2,C1,C2,C3,C4,XF,JTYPE,B,NLOP,GP,OMAG,DD)
      DIMENSION XF(10),B(20)
      WRITE            (3,1)
      HEXT=-.01
      HEXT=0.01
      HINT=0.
      GA=272.
      GB=141.5
      GR1=50.
      GR2=150.
      W1=0.0000782
      W2=.0475
      C1=3.890E 05
      C2=1.703E 06
      C3=4.883E 04
      C4=0.

C  INSERT COST SCALE FACTOR HERE
      DD=1.E-05
      C1=DD*C1
      C2=DD*C2
      C3=DD*C3
      C4=DD*C4
      GP=0.1
      OMG=GA*GR1*W1/W2
      NLOP=1
      XF(1)=0.
      XF(2)=0.
      XF(3)=0.
      XF(4)=0.
      GO TO(100,102),JTYPE
100   IF(HEXT)102,101,101
C  SAC
101   WRITE            (3,3)
      NDIM=6
      NDIMB=0
      NPUT=0
      GO TO 103
C  2 PT. OR FLOOD
102   NDIM=10
      NDIMB=8
      NPUT=2
      NPUT=1
      ERLMT=.001
103   WRITE            (3,2)NDIM,NDIMB,HEXT,HINT,GA,GB,W1,W2,C1,C2,
   1  C3,C4,GP,GR1,GR2,OMG,DD,ERLMT,NPUT
      WRITE            (3,9)XF(1),XF(2),XF(3),XF(4)
      GO TO(200,201),NLOP
200   WRITE            (3,4)
201   RETURN
    1 FORMAT(1H0,16X11HPROBLEM 3A.,/,17X97HOPTIMAL CONTROL OF A 4TH ORDE
   1R TYPE 2 REMOTE TRACKING SERVO WITH (NONLINEAR) 2 PHASE SERVOMOTOR
   2S.,/)
    2 FORMAT(1H0,17X5HNDIM=,I3,16X6HNDIMB=,I3,6X5HHEXT=,E14.7,
   1  6X5HHINT=,E14.7,/,20X3HGA=,E14.7,8X3HGB=,E14.7,8X3HW1=,E14.7,
   2  8X3HW2=,E14.7,/,20X3HC1=,E14.7,8X3HC2=,E14.7,8X3HC3=,E14.7,
   3  8X3HC4=,E14.7,/,20X3HGP=,E14.7,7X4HGR1=,E14.7,7X4HGR2=,E14.7,
   4  7X4HOMG=,E14.7,/,20X3HDD=,E14.7,5X6HERLMT=,E14.7,6X5HNPUT=,I2)
    3 FORMAT(1H ,16X24HSIMULATED ACTIVE CONTROL,/)
    4 FORMAT(1H0,/,17X23HLINEARIZED MODEL OPTION)
    9 FORMAT(1H ,16X6HXF(1)=,E14.7,5X6HXF(2)=,E14.7,5X6HXF(3)=,E14.7,
   1  5X6HXF(4)=,E14.7,/)
      END
      FUNCTION TORQUE(SPEED,VOLTS,NLOP)
      GO TO(100,101),NLOP
100   TORQUE= -.0000782*SPEED+.0475*VOLTS
      RETURN
```

```
  103 TORQUE=0.
      RETURN
      END
      SUBROUTINE INPUT(XI,HEXT,HINT,NFNL,NDAT,JTYPE,B,NLOP,THRD,THPR,
     1  W1,W2,GR2,GP)
      DIMENSION XI(10),B(20)
      GO TO(100,103),JTYPE
  100 IF(HEXT)102,101,101
C   SAC
  101 READ 4,B(1),B(2),B(3),B(4)
      READ 4,B(5),B(6),B(7),B(8)
      READ 4,XI(1),XI(2),XI(3),XI(4)
      READ 1,THRD,THPR,NFNL,NDAT
      WRITE               (3,5)B(1),B(2),B(3),B(4)
      WRITE               (3,6)B(5),B(6),B(7),B(8)
      XI(5)=0.
      XI(6)=0.
      GO TO 104
C   FLOOD
  102 READ 4,XI(5),XI(6),XI(7),XI(8)
      READ 1,THRD,THPR,NFNL,NDAT
      XI(1)=0.
      XI(2)=0.
      XI(3)=0.
      XI(4)=0.
      RNFNL=NFNL+1
      XI(9)=-RNFNL*HEXT-HINT
      XI(10)=0.
      GO TO 104
C   2 PT.
  103 READ 4,XI(1),XI(2),XI(3),XI(4)
      READ 4,XI(5),XI(6),XI(7),XI(8)
      READ 1,THRD,THPR,NFNL,NDAT
      XI(9)=0.
      XI(10)=0.
  104 GO TO(105,106),NLOP
  105 THPR=GR2*W1*THRD/(W2*GP)
  106 WRITE               (3,7)XI(1),XI(2),XI(3),XI(4)
      WRITE               (3,8)XI(5),XI(6),XI(7),XI(8)
      WRITE               (3,10)THRD,THPR,XI(9),XI(10)
      WRITE               (3,11)NFNL,NDAT
    1 FORMAT(2E14.7,2I4)
    4 FORMAT(4E14.7)
    5 FORMAT(1H0,17X5HB(1)=,E14.7,6X5HB(2)=,E14.7,6X5HB(3)=,E14.7,
     1  6X5HB(4)=,E14.7)
    6 FORMAT(1H ,17X5HB(5)=,E14.7,6X5HB(6)=,E14.7,6X5HB(7)=,E14.7,
     1  6X5HB(8)=,E14.7)
    7 FORMAT(1H0,16X6HXI(1)=,E14.7,5X6HXI(2)=,E14.7,5X6HXI(3)=,E14.7,
     1  5X6HXI(4)=,E14.7)
    8 FORMAT(1H ,16X6HXI(5)=,E14.7,5X6HXI(6)=,E14.7,5X6HXI(7)=,E14.7,

     1  5X6HXI(8)=,E14.7)
   10 FORMAT(1H ,17X5HTHRD=,E14.7,6X5HTHPR=,E14.7,5X6HXI(9)=,E14.7,
     1  4X7HXI(10)=,E14.7)
   11 FORMAT(1H ,17X5HNFNL=,I4,16X5HNDAT=,I4)
      END
      SUBROUTINE BOUNDR(LINK,NP,NPT,XI,XF,N,X,HF,HFD,MBTI,MBTF,PHF,
     1  C1,C2,C3,C4,GA,GB,OMG,GR2,GP,W1,W2,THRD,THPR,NLOP,DD)
      DIMENSION XI(10),XF(10),X(10,2,200),HF(10),HFD(10),MBTI(10)
      DIMENSION MBTF(10),PHF(10)
      IF(NP)101,100,101
C   BOUNDARY TRIGGER DECLARATION
  100 NP=4
      NPT=5
      NPT=4
      MBTI(1)=5
      MBTI(2)=6
      MBTI(3)=7
      MBTI(4)=8
      MBTF(1)=1
      MBTF(2)=2
      MBTF(3)=3
      MBTF(4)=4
  101 GO TO(102,104),LINK
C   G=0 (INITIAL CONDITIONS)
  102 DO 103 I=1,10
  103 X(I,1,N)=XI(I)
      RETURN
C   H=0 (TERMINAL CONDITIONS)
  104 HF(1)=X(1,1,N)-XF(1)
      HF(2)=X(2,1,N)-XF(2)
      HF(3)=X(3,1,N)-XF(3)
      HF(4)=X(4,1,N)-XF(4)
      IF(NPT-NP)106,106,105
  105 HF(5)=-(C1*X(1,1,N)*X(1,1,N)+C2*X(2,1,N)*X(2,1,N)+C3*X(3,1,N)*
     1  X(3,1,N)+C4*X(4,1,N)*X(4,1,N)-GA*GA*X(8,1,N)*X(8,1,N)/(4.*DD))
     2  +X(2,1,N)*X(5,1,N)+GB*X(6,1,N)*2.*TORQUE(GR2*(THRD+X(2,1,N)),
     3  GP*(THPR+X(3,1,N)),NLOP)+X(4,1,N)*X(7,1,N)-OMG*X(4,1,N)*X(8,1,N)
      HFD(1)=X(2,1,N)
      HFD(2)=2.*GB*TORQUE(GR2*(THRD+X(2,1,N)),GP*(THPR+X(3,1,N)),NLOP)
      HFD(3)=X(4,1,N)
      HFD(4)=-OMG*X(4,1,N)+GA*GA*X(8,1,N)/(2.*DD)
      HFD(5)=0.
      PHF(1)=-2.*C1*X(1,1,N)
      PHF(2)=-2.*C2*X(2,1,N)+X(5,1,N)-2.*W1*GR2*GB*X(6,1,N)
      PHF(3)=-2.*C3*X(3,1,N)+2.*GP*W2*GB*X(6,1,N)
      PHF(4)=-2.*C4*X(4,1,N)+X(7,1,N)-OMG*X(8,1,N)
      PHF(5)=X(2,1,N)
      PHF(6)=GB*2.*TORQUE(GR2*(THRD+X(2,1,N)),GP*(THPR+X(3,1,N)),NLOP)
      PHF(7)=X(4,1,N)
      PHF(8)=GA*GA*X(8,1,N)/(2.*DD)-OMG*X(4,1,N)
```

```
  106 RETURN
      END
      SUBROUTINE DFEQNS(DX,DXD,J,X,N,GA,GB,GR1,GR2,W1,W2,C1,C2,C3,C4,
     1  GP,HEXT,HINT,B,THRD,THPR,OMG,NLOP,JTYPE,DD)
      DIMENSION DX(10),DXD(10),X(10,2,200),B(20),F(20)
      GO TO (99,101),J
   99 GO TO(113,100),JTYPE
  113 IF(HEXT)100,105,105
  100 GO TO(500,501),NLOP
C  FORWARD 2 PT. OR FLOOD
  500 PTX2=-2.*GR2*W1
      PTX3=2.*W2*GP
      GO TO 502
  501 PTX2=2.*(TORQUE(GR2*(THRD+10.5+DX(2)),GP*(THPR+DX(3)),NLOP)
     1  -TORQUE(GR2*(THRD+DX(2)),GP*(THPR+DX(3)),NLOP))/10.5
      PTX3=2.*(TORQUE(GR2*(THRD+DX(2)),GP*(THPR+DX(3)+1.),NLOP)
     1  -TORQUE(GR2*(THRD+DX(2)),GP*(THPR+DX(3)),NLOP))
  502 U=GA*DX(8)/(2.*DD)
      DXD(1)=DX(2)
      DXD(2)=GB*2.*TORQUE(GR2*(THRD+DX(2)),GP*(THPR+DX(3)),NLOP)
      DXD(3)=DX(4)
      DXD(4)=-OMG*DX(4)+GA*U
      DXD(5)=2.*C1*DX(1)
      DXD(6)=2.*C2*DX(2)-DX(5)-GB*DX(6)*PTX2
      DXD(7)=2.*C3*DX(3)-GB*DX(6)*PTX3
      DXD(8)=2.*C4*DX(4)-DX(7)+OMG*DX(8)
      DXD(9)=1.
      DXD(10)=C1*DX(1)*DX(1)+C2*DX(2)*DX(2)+C3*DX(3)*DX(3)+
     1  C4*DX(4)*DX(4)+DD*U*U
      RETURN
C  BACKWARD 2 PT. (ADJOINT)
C  LINEARIZED ADJOINT EQNS. ONLY (NLOP=1)
  101 DXD(1)=-2.*C1*DX(5)
      DXD(2)=2.*GB*GR2*W1*DX(2)-2.*C2*DX(6)-DX(1)
      DXD(3)=-2.*GB*W2*GP*DX(2)-2.*C3*DX(7)
      DXD(4)=-DX(3)+OMG*DX(4)-2.*C4*DX(8)
      DXD(5)=DX(6)
      DXD(6)=-2.*GB*(GR2*W1*DX(6)-W2*GP*DX(7))
      DXD(7)=DX(8)
      DXD(8)=-GA*GA*DX(4)/(2.*DD)-OMG*DX(8)
      RETURN
C  SAC
  105 DXD(1)=DX(2)
C  INSERT TERMS IN CONTROL LAW U=U(X) HERE
      MNT=4
      F(1)=DX(1)
      F(2)=DX(2)
      F(3)=DX(3)
      F(4)=DX(4)
      U=0.

      DO 108 I=1,MNT
  108 U=U+B(I)*F(I) '
      DXD(2)=GB*2.*TORQUE(GR2*(THRD+DX(2)),GP*(THPR+DX(3)),NLOP)
      DXD(3)=DX(4)
      DXD(4)=-OMG*DX(4)+GA*U
      DXD(5)=1.
      DXD(6)=C1*DX(1)*DX(1)+C2*DX(2)*DX(2)+C3*DX(3)*DX(3)+
     1  C4*DX(4)*DX(4)+DD*U*U
      RETURN
      END
      SUBROUTINE AUXDTA(X,NFNL,HEXT,HINT,U1,T1,T2,GA,GB,GR1,GR2,W1,W2,
     1  C1,C2,C3,C4,GP,B,THRD,THPR,NLOP,JTYPE,NDAT,NPOINT,RMSER,NLAST,
     2  DD)
      DIMENSION X(10,2,200),U1(200),T1(200),T2(200),B(20),F(20)
      IF(HINT)111,110,111
  110 NLAST=NFNL
      GO TO 112
  111 NLAST=NFNL+1
  112 N=1-NDAT
      NPOINT=0
  106 N=N+NDAT
   98 NPOINT=NPOINT+1
      GO TO(105,205),JTYPE
  105 IF(HEXT)205,135,135
C  2 PT. OR FLOOD
  205 U1(N)=GA*X(8,1,N)/(2.*DD)
      GO TO 317
C  SAC
  135 U1(N)=0.
C  INSERT TERMS IN CONTROL LAW U=U(X) HERE
      MNT=4
      F(1)=X(1,1,N)
      F(2)=X(2,1,N)
      F(3)=X(3,1,N)
      F(4)=X(4,1,N)
      DO 96 I=1,MNT
   96 U1(N)=U1(N)+B(I)*F(I)
  317 T1(N)=TORQUE(GR1*X(4,1,N),U1(N),1)
      T2(N)=2.*TORQUE(GR2*(THRD+X(2,1,N)),GP*(THPR+X(3,1,N)),NLOP)
  100 IF(N+NDAT-NLAST)106,106,99
   99 IF(N-NLAST)102,103,102
  102 N=NLAST
      GO TO 98
  103 GO TO(400,401),JTYPE
  400 IF(HEXT)401,401,402
  402 RMSER=SQRT (X(1,1,N)*X(1,1,N)+X(2,1,N)*X(2,1,N)+X(3,1,N)*X(3,1,N)
     1  +X(4,1,N)*X(4,1,N))
  401 RETURN
      END
      SUBROUTINE OUTPUT(X,NFNL,U1,T1,T2,MOUT,NPUT,MRUN,NPOINT,NDAT,
```

```
      1  NLAST,JTYPE,HEXT)
       DIMENSION X(10,2,200),U1(200),T1(200),T2(200)
       WRITE            (3,3)MRUN,NPOINT,NPUT
       GO TO(220,222),JTYPE
   220 IF(HEXT)222,221,221
   221 WRITE            (3,1)
   222 N=1-NDAT
   200 N=N+NDAT
   199 GO TO(201,202),JTYPE
   201 IF(HEXT)202,203,203
C  2 PT. OR FLOOD
   202 WRITE            (3,6)N,X(9,1,N),X(1,1,N),X(2,1,N),X(3,1,N),
      1  X(4,1,N),X(10,1,N),X(5,1,N),X(6,1,N),X(7,1,N),X(8,1,N),U1(N),
      2  T1(N),T2(N)
       GO TO(204,208),NPUT
   208 GO TO(209,210),MOUT
   209 WRITE      (5)MRUN,N,X(5,1,N),X(1,1,N),X(2,1,N),X(3,1,N),
      1  X(4,1,N),U1(N),T1(N),T2(N)
       GO TO 204
   210 WRITE            (3,4)
       NPUT=1
       GO TO 204
C  SAC
   203 WRITE            (3,2)N,X(5,1,N),X(1,1,N),X(2,1,N),X(3,1,N),
      1  X(4,1,N),X(6,1,N),U1(N),T1(N),T2(N)
   204 IF(N+NDAT-NLAST)200,200,205
   205 IF(N-NLAST)206,207,206
   206 N=NLAST
       GO TO 199
   207 GO TO(211,213),JTYPE
   211 IF(HEXT)212,215,215
   212 N=NLAST
       GO TO 214
   213 N=1
   214 WRITE            (3,5)X(5,1,N),X(6,1,N),X(7,1,N),X(8,1,N)
   215 RETURN
     1 FORMAT(1H0,2H N,5X4HTIME,7X8HX(1,1,N),5X8HX(2,1,N),5X8HX(3,1,N),
      1  5X8HX(4,1,N),7X2HIP,10X5HU1(N),8X5HT1(N),8X5HT2(N),/)
     2 FORMAT(1H ,I4,1XE10.4,1XE12.6,1XE12.6,1XE12.6,1XE12.6,1XE12.6,
      1  1XE12.6,1XE12.6,1XE12.6)
     3 FORMAT(1H0,17X14HTRAJECTORY NO.,I3,1H.,I5,19H DATA POINTS. NPUT=,
      1  I2)
     4 FORMAT(1H0,39H***NO TAPE WRITE PERMITTED WITH MOUT=2.)
     5 FORMAT(1H0,17X25HCO-STATE STARTING VECTOR=,2XE14.7,2XE14.7,
      1  2XE14.7,2XE14.7)
     6 FORMAT(1H ,2HN=,I5,3X3H T=,E14.7,3X3HX1=,E14.7,3X3HX2=,E14.7,
      1  3X3HX3=,E14.7,3X3HX4=,E14.7,/,11X3HIP=,E14.7,3X3HX5=,E14.7,
      2  3X3HX6=,E14.7,3X3HX7=,E14.7,3X3HX8=,E14.7,/11X3HU1=,E14.7,
      3  3X3HT1=,E14.7,3X3HT2=,E14.7,/)
       END
```

APPENDIX E

SELECTION OF EFFICIENT ESTIMATION VARIABLES $Z_i(x,t)$

For selected terms $Z_i(x,t)$, often called estimation variables, least squares determines the coefficients k_i of the closed-loop function

$$\hat{u} = \sum^{r} k_i Z_i(x,t) \tag{3.1}$$

The wide choice of linearly independent Z_i functions burdens the selection, however, since results may depend critically on the choice. Moreover, many otherwise suitable candidates are unacceptable on the basis of overriding hardware constraints. Hardware constraints aside, the ideal selection consists of only one term, i.e., let $Z_i = u^*(x,t)$. The ideal begs the question, however, if u^* is known only as a set of tabulated data. Graphical displays are suggested to inspire an effective choice of estimation variables in curve fitting problems. Graphical inspiration becomes clouded, however, with the multivariate hypersurfaces of the typical $u^*(x,t)$.

In view of the difficulties, a widely used approach is to seek an efficient means for selecting combinations of estimation variables from a larger pool of candidates, i.e., selection of the best r out of p terms Z_i, where $p \gg r$. This approach readily allows external factors such as hardware constraints and knowledge and experience with similar cases to be incorporated. Thus, only those linearly independent terms deemed desirable, and possible suspected as effective, need be contained in the candidate set.

If the size of the candidate pool is even moderately large, the possibility of testing all L combinations of r out of p terms may become impractical.[1] For example, with only 9 candidate terms in the pool, testing for the best combination of 4 requires a total of 126 tests. If the best 4 are deficient, further testing for the best 5 in 9 costs an added 126 exercises. Complete testing for the best 5 from a larger pool of 50 would lead to 46,060 fittings.

Thus, in devising a worthwhile scheme the crucial measure of success is the ability to arrive at a reasonably efficient selection from a relatively small number of tests. The method most widely discussed in the literature for selecting the best r out of p estimation variables in a least squares problem[2] makes use of the so-called step-up procedure. References 36 and 57 give excellent discussions of the step-up procedure, and present computer implementations. The former report includes some experimental verification on several data sets generated from algebraic functions. A brief heuristic description of the step-up procedure is given below. The computer program of Ref. 36 is then further applied to the open-loop control data of the Case One study to demonstrate the performance of the method.

---

[1]The appropriate relation is $L = \frac{p!}{r!(p - r)!}$

[2]Ultimate suitability is resolved only by simulation runs. Here least squares acts to select the most promising combinations for extensive simulation.

Step-Up Procedure

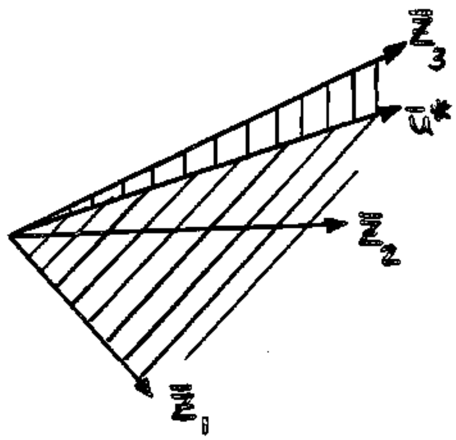
An estimation variable Z_i is said to be activated when it is selected from the candidate set for use in the control function $\hat{u}$. The step-up procedure seeks to construct an efficient equation through a step by step activation of individual terms from those remaining in the inactive pool. Thus, the initial selection consists of the term which offers the least sum of squares of error. Each subsequent selection consists of that term which combines with those in the activated set to give the greatest further reduction in sum of squares of error. The process is stopped when the total number of desired terms have been activated, or when other possible stopping rules intervene, as discussed below.

In this manner, the L tests required to select r out of p variables is far less than testing all possible combinations.[1] Thus, selection of 4 terms from a pool of 9 calls for 30 tests (versus 126 possible combinations). Moreover, if 4 terms are insufficient, selecting 5 out of 9 calls for 30 tests (versus 126 possible combinations). Even a selection of 5 terms in 50 is achieved with relatively few comparisons (240 versus 46,060 combinations). Equally important, the tests for the step-up procedure can be readily derived from intermediate results used in the least squares calculations. Hence, computer time comparisons for a step-up procedure in a least squares problem tend to be highly favorable over that expected for testing all possible combinations.

---

1 With a step-up, $L = \sum_{k=0}^{r-1} (p-k)$

While experience has indicated that the step-up procedure is an effective means of selecting efficient estimation variables, the resulting choice is not always strictly optimal. This is perhaps best understood by viewing the collection of data points u_j^* as a vector $\overline{u}$,* and similarly each $Z_i(x_j,t_j)$ forms a vector of data $\overline{Z}_i$ in E^M (M data points). Now suppose that among three estimation vectors $(\overline{Z}_1,\overline{Z}_2,\overline{Z}_3)$ the dependent data vector $\overline{u}^*$ lies in the plane of the first two, but actually closer to $\overline{Z}_3$ (see sketch below).



In operation, the step-up procedure would begin by activating Z_3 (whose data vector most closely coincides with $\overline{u}^*$) followed by an activation of one of the others. Yet this combination of two is less effective than the two vectors complanar with $\overline{u}^*$.

To alleviate this defect in a simple step-up approach, the procedure can be augmented by rules for deactivating the least efficient estimation variable from the active set. Thus, consider again the above example. The best two estimation variables could be discovered after the third step by deactivating that variable which gave least improvement to the remaining active subset (i.e., delete Z_3 to retain the strongest pair Z_1, Z_2). Various <u>throw-out criteria</u>, discussed below, have been suggested for deactivation decision rules. Despite apparent success with step-up/throw-out combinations, examples can yet be found where selections are nonoptimal (though nearly so) in the sense of least squares.[1]

Various algorithms for implementing step-up procedures seem largely distinguished by the actual combinations of stopping rules and throw-out criteria employed. The program of Ref. 36 allows for a number of options. Stopping rules include reaching a minimum level of multiple correlation (R), a test for round off error in the least squares calculations, and selection of a prespecified number of terms. A throw-out option repeats the decision rule: activate two variables and throw out one.

The F statistic[2] is suggested and employed in stopping and throw-out decision rules in References 36, 57, and 58. Computationally, F is the ratio

$$F = \frac{MSE}{MSR} \tag{E-1}$$

where

$$MSE = \frac{SSE}{DFE} \quad \text{(mean square error)}$$

$$MSR = \frac{(SSD - SSE)}{DFR} \quad \text{(mean square error of regression)}$$

$$SSE = \sum_{j}^{M} (\hat{u}_j - u_j^*)^2 \quad \text{(sum of squares of error)}$$

---

[1]Ref. 36, section titled Empirical Computer Studies.

[2]See e.g., Ref. 59, pp. 140.

$$SSD = \sum_{j}^{M} (u_j^* - u_o^*)^2 \quad \text{(sum of squares of deviations)}$$

$$DFE = M - r - 1 \quad \text{(degrees of freedom for error)}$$

$$DFR = r \quad \text{(degrees of freedom for regression)}$$

$$u_o^* = \sum_{j}^{M} u_j^*/M \quad \text{(mean)}$$

M = no. of data points

r = no. of active estimation variables

The F ratio is a well known tool for hypothesis testing in statistical decision theory. By comparing the F value with tabulated critical values one establishes, within a specified degree of confidence and subject to the assumptions of the theory, statistical significance (or its absence) of a given effect over that of random chance.[1]

As applied to step-up procedures the F ratio can be used heuristically for testing the significance of adding (deleting) an estimation variable to (from) the active set. Higher values indicate greater significance. Thus, the following decision rules are options in the program of Ref. 36:

1. Stop if the F value of the last activated variable is below a threshold level F_I.
2. Throw out a variable from the active set if its F value is below a threshold level F_o.

---

[1]Tables VI(a) and VI(b) of Ref. 59 give tabulated values, under varying degrees of freedom, at the 5 and 1 percent confidence levels.

Experimental Results

The simulation results of Figures 8, 9 (first case study, Chapter IV) were obtained from control functions derived by the step-up computer program of Ref. 36. No throw-out criteria was employed. In the present section the performance of the step-up/throw-out approach is demonstrated by extending the applications of the Case One study and comparing these results with those obtained in Chapter IV.

Performance in selecting efficient estimation variables is examined under a wide range of test conditions. A total of 154 simulation runs are involved, giving, it is believed, a reasonable base of information for examining the approach in the Case One problem.

Figures 38, 39, 40, and 41 show terminal error figures at zero time-to-go for control functions selected by the step-up procedure. Comparisons are made both with and without the benefit of an F statistic throw-out criteria.

Figure 38 is an application to the 130 point, widely spread (Fig. 6) data base selection. In this case the step-up, operating from the least squares criterion, appears to give reasonably good results in terms of terminal error simulation performance. The 4, 5 term selections are essentially as good as those of further additions. The pattern of results is similar for both initial conditions. The throw-out decision rules affect only the 7-9 term region, where changes make little difference.

Added confidence is gained with a similar data base for the long response time problem (T = 15). Results for this case are given in Figure 8 of Chapter IV, where the two term selection was x_1, x_2 (as

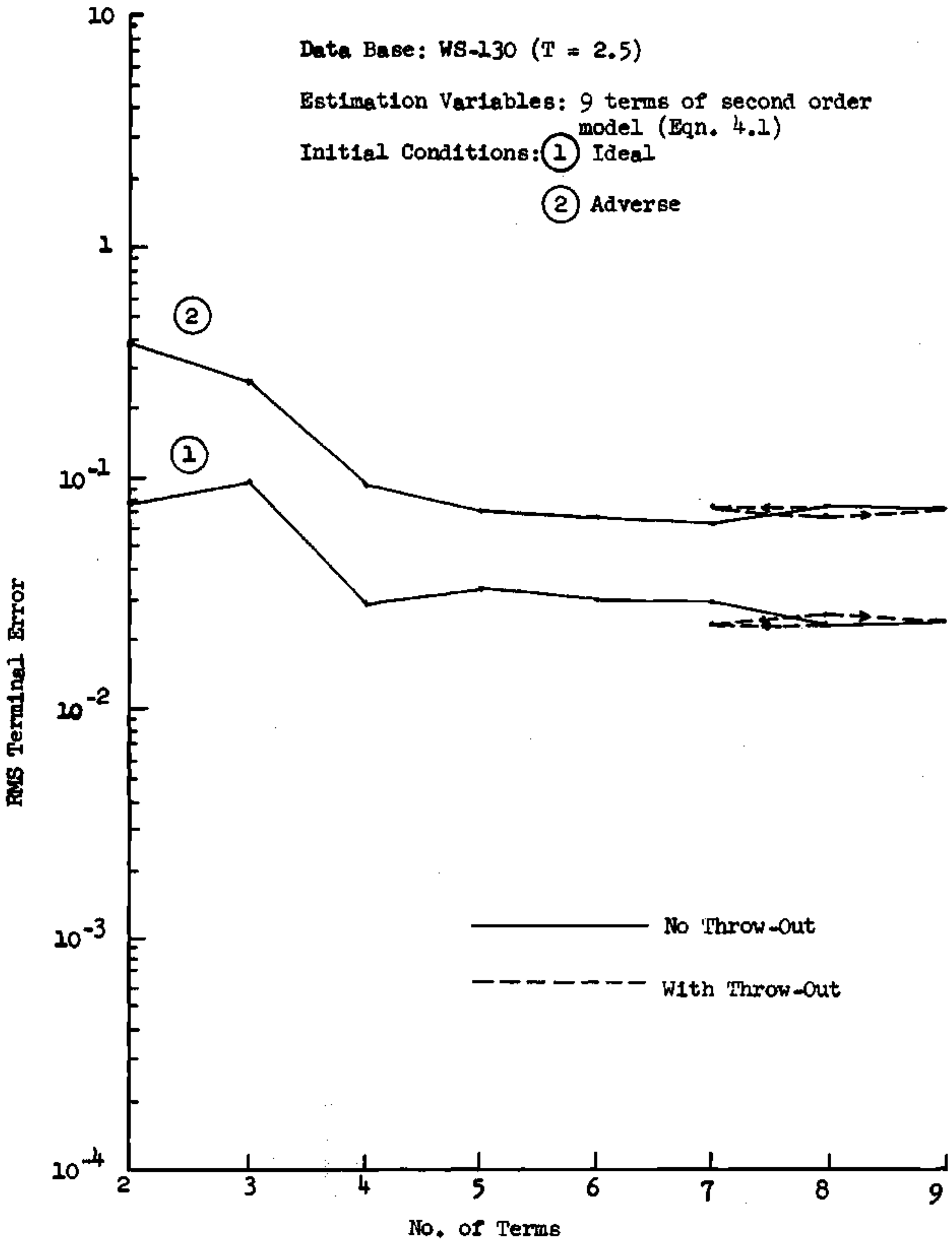


Fig. 38 Step-Up Performance: Wide Base, Fewer Points

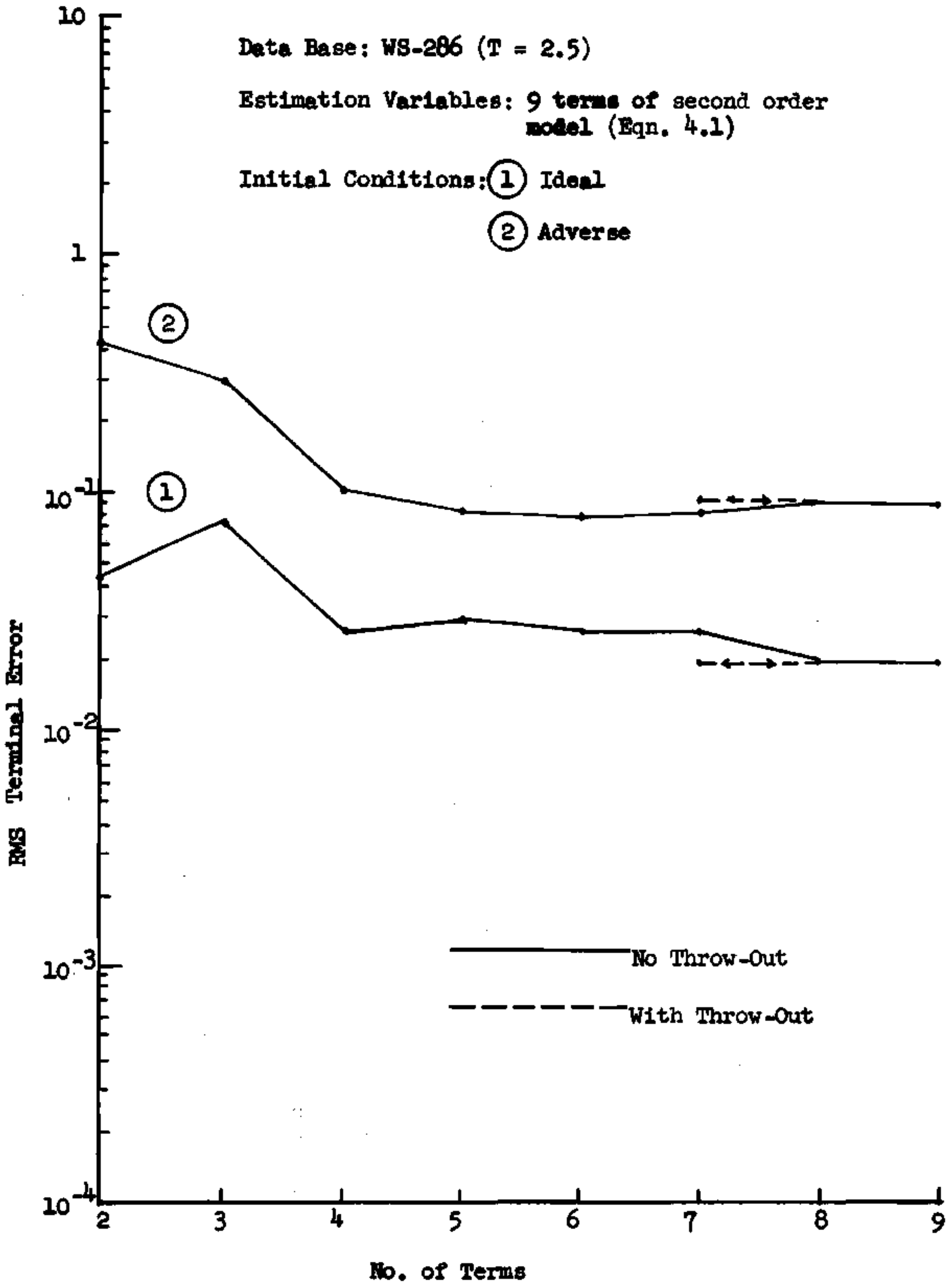


Fig. 39 Step-Up Performance: Wide Base More Points

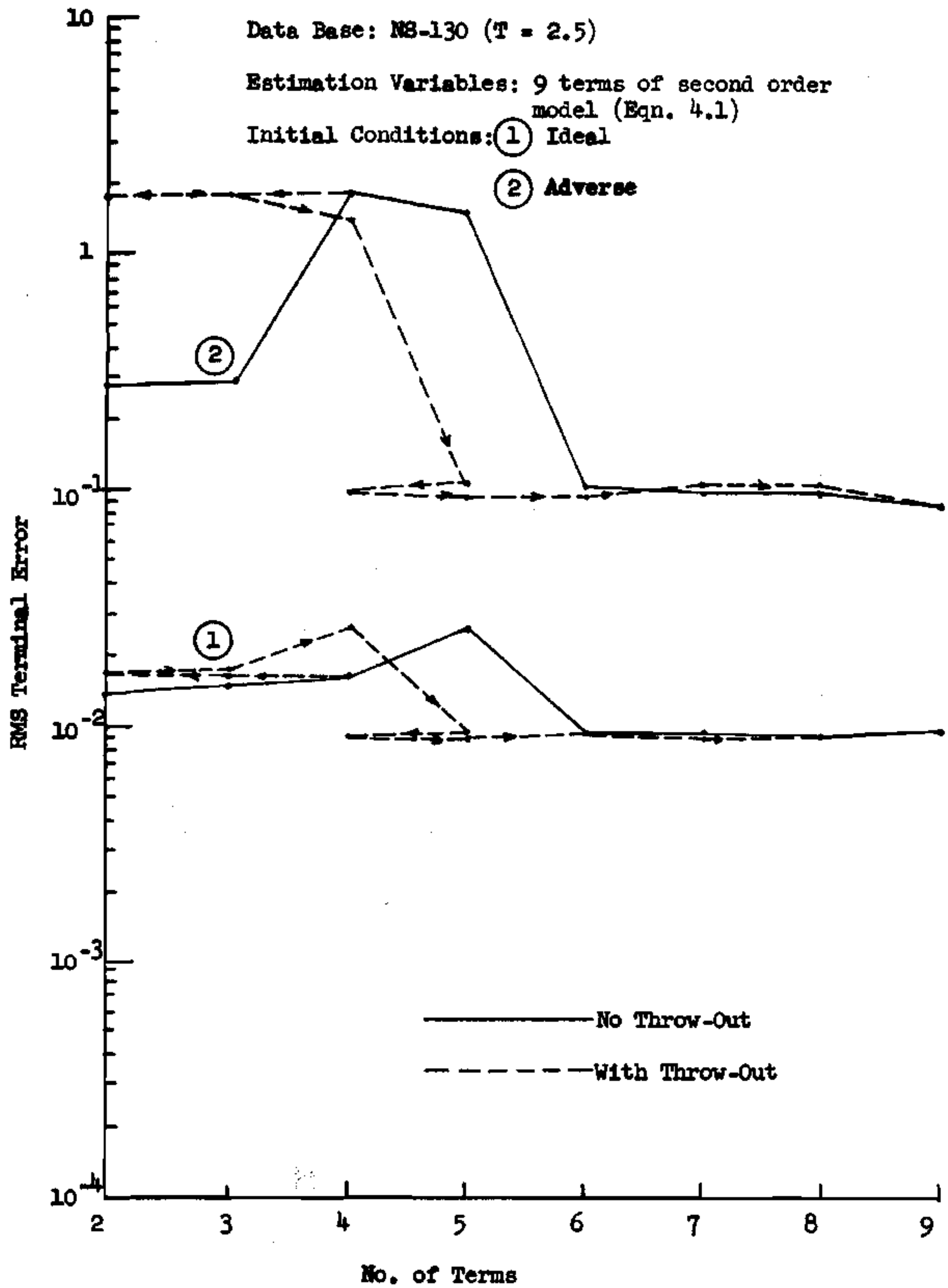


Fig. 40 Step-Up Performance: Narrow Base, Fewer Points

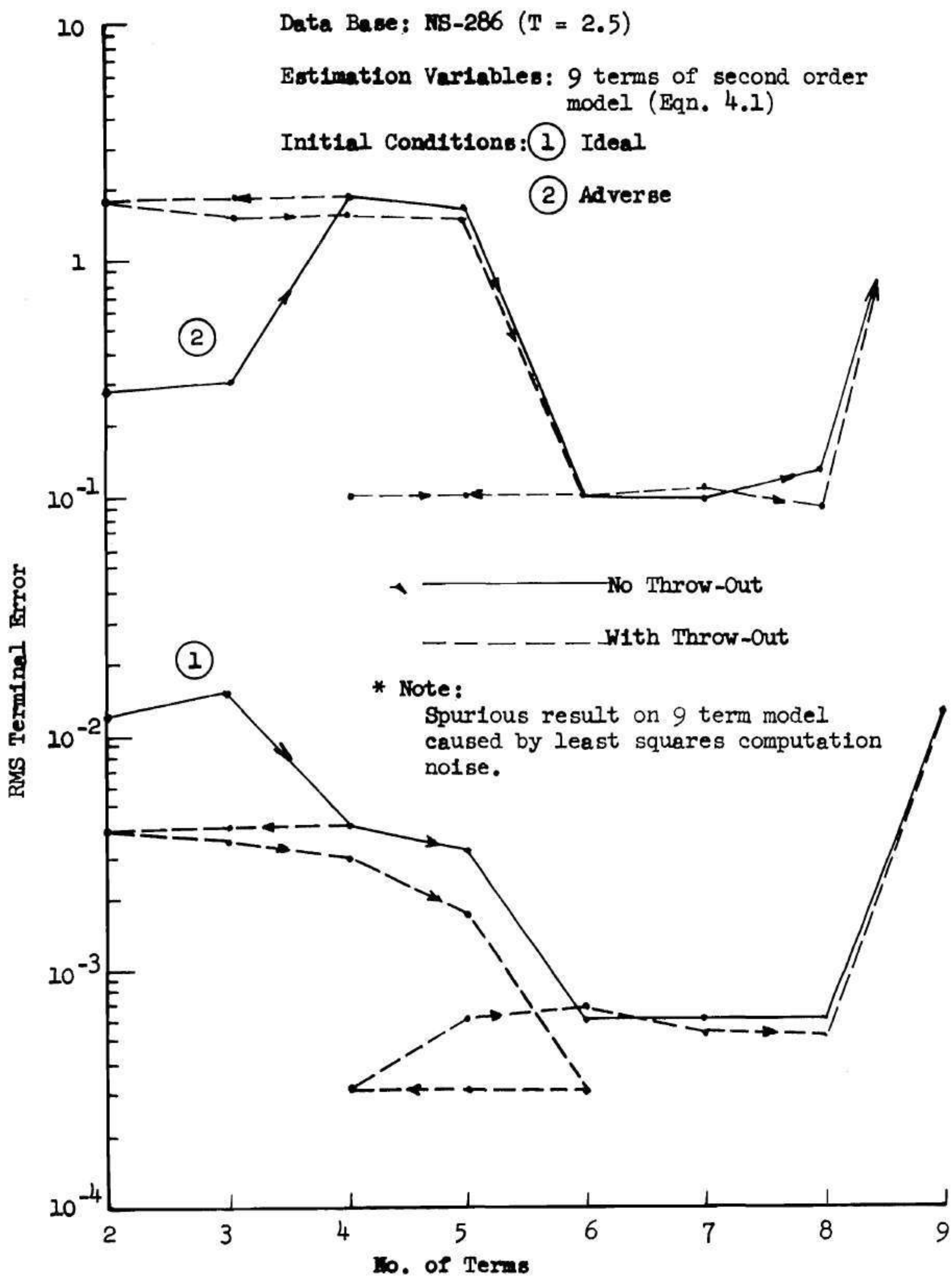


Fig. 41 Step-Up Performance: Narrow Base, More Points

expected, in close agreement with closed form results for the case $T \rightarrow \infty$). Further results for the data base of Figure 38 are available from Figure 9 (Chapter IV). Estimation variables are selected from a time varying gain form (eqn. 4.2). The latter set, more appropriate to the application, achieves a generally improving performance trend from the step-up. The throw-out criteria (not shown) again had scant effect.

Figure 39 is similar to Figure 38 with one exception: the number of data points was more than doubled to 286. The change had essentially no effect on the step-up performance.

Figures 40, 41 repeat the test condition patterns of Figures 38, 39 except that a narrow data base spread is used (Fig. 6). As shown, this change provoked early action by the throw-out criteria, with substantial benefit to the simulation results.

One spurious result appears with the full nine term equation in Figure 41. A check revealed that computation noise lead to errors in the least squares coefficients. Further spot checks showed the noise problem to be nil in those cases examined. Double precision calculations could help, but with single precision only one spurious result was detectable from the entire group. Note that the noise problem was associated with the least squares solution itself, and was not a fault of the step-up procedure.

The data base of Figure 41 also gave a satisfactory step-up performance trend when fitted with estimation variables from the time varying gain form (eqn. 4.2). The results are shown in Figure 9. In contrast with the less efficient variables from the second order model, the effect of the throw-out criteria (not shown) with the TVG variables was minimal.

Conclusions

The problem of selecting efficient estimation variables Z_i has been examined. As evidenced by the literature, a widely practiced approach is to phrase the problem as one of selecting the best r terms out of a larger pool p. An attractive feature is that overriding hardware constraints and a priori knowledge of the problem are readily integrated in the selection.

The step-up procedure offers a potentially efficient means to conduct the selection process. A heuristic description of the method is given, along with a discussion of commonly used decision criteria for augmenting the basic procedure.

Performance of the method is demonstrated on the open-loop data of the first case problem, using the implementation of Ref. 36. The method, operating from a least squares criteria, appeared to give efficient selections for a fixed number of terms, when compared on the basis of closed-loop performance. This generally held true under varying conditions of data base, type of estimation variables in the pool, and simulation initial conditions. In some cases the F statistic throw-out criteria was found to substantially improve the selections over a simple step-up. Greatest effectiveness appeared to occur where it was needed most, i.e., with the poorest available choice of estimation variables and the narrowest dispersions of data.

VITA

David Eric Peterson was born in West Chester, Pennsylvania on August 13, 1932 and was moved soon afterward by his parents to live in Milwaukee, Wisconsin. He obtained his B.S. degree in Mechanical Engineering from Purdue University in 1954. He has acquired a variety of technical and supervisory experience during seven years of industrial employment with the Heil Company, AC Electronics Division of General Motors, and the Astronautics Corporation of America. His military obligation was served with the U. S. Army during 1955-1957. Mr. Peterson is a registered Professional Engineer in the State of Wisconsin. While working full-time, he pursued a Master's Degree on a part-time basis with the University of Wisconsin, completing his M.S.M.E. program in 1963. He later entered the Georgia Institute of Technology to work toward the degree of Doctor of Philosophy in the School of Mechanical Engineering.

Mr. Peterson was married for six years to the late and former Mary Emmeline Duncan of Atlanta, Georgia.